Foundations of Cosmology

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Cosmology: Study of the Entire Universe, Its Origin, Evolution, & Ultimate Fate

Genesis 1:3 "*And God said, Let there be light: and there was light.*"
Gen 15: 5 God said *"Look toward heaven, and number the stars."*
Psalms 19:1 "*The heavens declare the glory of God.*"

Purpose:

This paper was done to familiarize myself with some of the basic concepts of the Theories of Cosmology. Some of the original papers, data, math, and concepts of the Big Bang Lambda Cold Dark Matter Model (ΛCDM) were reviewed. **This paper is not original work.** These concepts were then used to evaluate the models. The evidence and concordance for the various models are shown in the various plots of model parameters. **One goal was to capture the concepts and mathematical models of Cosmology in a Functional Type Programming paradigm such as Mathcad, that closely follows the traditional mathematical notation presented in the format of a worksheet. Bottom Line: To make the math reasoning, logic, and the programming explicit.**

Mathcad operations are shown in purple italics. For example: $sin\left(\frac{\pi}{2}\right) = 1$

"The popular notion that the sciences are bodies of established fact is entirely mistaken. Nothing in science is permanently established, nothing unalterable, and indeed science is quite clearly changing all the time, and not through the accretion of new certainties." Karl Popper Particularly applicable to revisions of ΛCDM Model needed to match JWST observations.

"The progress of science is strewn, like an ancient desert trail, with the bleached skeletons of discarded theories which once seemed to possess eternal life." Arthur Koestler

Two Contrasting World Views Concerning the Validity of Big Bang Singularity Hypothesis

Three Evidences for the ΛCDM Model: The expansion of the universe according to Hubble's law (as indicated by the redshifts of galaxies), the discovery and measurement of the Cosmic Microwave Background Radiation (CMBR), and the relative abundances of light elements produced by Big Bang nucleosynthesis.

***An Introduction To Modern Cosmology*, Andrew Liddle**

"The development of cosmology will no doubt be seen as one of the scientific triumphs of the twentieth century. At its beginning, cosmology hardly existed as a scientific discipline. By its end, the Hot Big Bang cosmology stood secure as the accepted description of the Universe as a whole. The turn of the millennium saw the establishment of what has come to be known as the Standard Cosmological Model, representing an almost universal consensus amongst cosmologists as to the best description of our Universe,"

***Dismantling the Big Bang, Reasons Why to Reject the Big-Bang Theory*, Alex Williams, J. Hartnett**

The theory lacks a credible and consistent mechanism.

"The big-bang universe begins in a singularity (entire universe crushed into a point of infinite density) and there is no known mechanism to start the universe expanding out of the singularity — the equations in the theory only work after the expansion has begun.It then requires a hypothetical period of stupendous inflation and stopping at a precise point to halt the universe from recollapsing. It further requires incredible fine tuning to maintain stability. Its mechanism for turning primordial energy into matter would produce equal amounts of matter and anti-matter but our universe is made only of matter. It has to violate physical laws and appeal to unknown forces (dark energy) and substances (dark matter) to explain what we observe. It is inconsistent with Thermodynamcis. It cannot explain the low entropy at the initial expansion." The detailed particle physics mechanism responsible for inflation is not known.

Science cannot produce any final answers on the subject of origins.

Science works in the present, by observation and experiment; it has no direct access to the past. We cannot directly observe the past, we cannot revisit it in a time machine, nor can we repeat it (as an experiment would require), so anything scientists say about the past has to be based on extrapolation from present-day observations. These extrapolations have, in turn, to be based on assumptions. Those assumptions are necessarily constructed within the framework of a belief system about the nature of the universe and how it came to be the way it is."

April 22, 2024

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Unspoken Assumptions

Most people today believe because they have been taught it is so, that physics can explain the Origin of the universe. This is the Assumptions of Naturalism: The idea that matter is all there is. The current orthodoxy of cosmology rests on a number of unexamined assumptions that have huge implications for our view of the universe.

"***The first principle is not to fool yourself – and you are the easiest person to fool.***" Richard Feynmann

What is Science? The Era of Post Empirical Science

The philosopher Karl Popper argued that what distinguishes a scientific theory from pseudoscience and pure metaphysics is the possibility that it might be falsified on exposure to empirical data.
In other words, a theory is scientific only if it has the potential to be proved wrong.
We live in the era of post empirical science. Major Concepts such as a Multiverse, Bouncing Universe, or 24 Dimensional String Theory can never be falsified. By Popper's criteria, these concepts do not constitute areas of legitimate scientific inquiry. One can hold the position that String Theory is manifestly false. It fails all predictions.

Limits of Science:

Science can only study things that happen more than once. By this definition, many areas of Cosmology can never be verified or falsified. These areas would be in the **realm of speculati**on. To explain something means to describe the unknown in terms of the known. **Unknown concepts such as Dark Energy or Cold Dark Matter do not do well to further elucidate area of inquiry.** They are **more in the arena of Metaphysics rather than Physics.**

The deeper we look into Physics and also the Biological Structure of the human cell, the deeper we see into a perhaps never ending depth of complexity.

"*Time and again the passion for understanding has led to the illusion that man is able to comprehend the objective world rationally by pure thought without any empirical foundations – in short, by metaphysic*s."
Albert Einstein

Distinguishing Between the Realms of Myth, Philosophy, Explanations, Metaphysics, and Science

One case where science crosses over into religion is the beginning of our universe. Physicists have put forward many theories for it: a big bang, a big bounce, a collision of higher-dimensional membranes, a gas of strings, a network, a 5-dimensional black hole, and many more. But the scientifically correct answer is, that we don't know how the universe began. There are good reasons to think we will never know. A greater cause that transcends the physical may be the origin. But many are unwilling to accept this possibility. Many fill this knowledge gap with tall tail creation myths, written in the language of Mathematics, such as a landscape of multiverses populated with googles of string topologies.

Perceived Tension Between Cosmology and Christianity - The Origin, Purpose, Destiny of the Universe

Christianity is consistent with Cosmology. Genesis 1:3 And God said "*Let there be light*" and there was light.
Creation of the universe Ex Nihilo with an explosion of light. Chronologically, the different concepts men have had about the nature of the universe were: first Pagan God Centered, then Earth Centered, then Sun Centered, then The Great Enlightenment Material Centered, and presently, the latest Science of Cosmology Centered. With respect to origin, what is the greatest possible cause, the Universe or God? With respect to these, what is the greatest possible explanation for our reality: Explanations for the physical properties and interactions of matter/energy or a Transcendent Being? Which is the more Transcendent origin? Our Perceptions and Theories about the Physical or God? The Law of Cause and Effect: The cause must always be greater than the effect. Who or What can create time?
Which is the more real? Abstract Field Theories about Matter, Tensor Field Equations, and Singularities or the Spiritual Influence of the God who is the Creator of the Laws of Physics and Math? Philosophically it is God.
In the end, God is the creator of the Laws of Physics and Mathematics. He is the Great Lawgiver of the Universe.
All truth, including the physical, is God's truth. There is no tension between Material Cosmology and the Biblical Old and New Testaments as long as the Biblical Account of Creation is not interpreted out of context.

IIA. The ΛCDM or Lambda-CDM Concordance Model of Cosmology

The ΛCDM or Lambda-Cold Dark Matter Model is a **parameterization of the Big Bang cosmological model** in which the universe contains three major components: first, a cosmological constant denoted by Lambda associated with dark energy; second, the postulated cold dark matter; and third, ordinary matter. **A Concordance cosmology is a model of the universe that assumes a minimum number of parameters**, especially the Lambda-CDM model, which has 6 parameters: physical baryon density parameter; physical dark matter density parameter; the age of the universe; scalar spectral index; curvature fluctuation amplitude; and reionization optical depth. Different sorts of measurements — each using different kinds of instruments to look at completely different kinds of objects, all involving different kinds of physical processes, give completely consistent results. It is frequently referred to as the Standard Model of Big Bang Cosmology because it is **the simplest model that provides a reasonably good account** of the following properties of the cosmos:

- the existence, structure, uniformity, and magnitudes of **anisotropies of the cosmic microwave background**
- the large-scale structure in the **distribution of galaxies**
- the **observed abundances** of hydrogen (including deuterium), helium, and lithium
- **the accelerating expansion of the universe** observed in the light from distant galaxies and supernovae

This model assumes that General Relativity (GR) is the correct theory of gravity on cosmological scales. It emerged in late the 1990s as a concordance cosmology, after a period of time when disparate observed properties of the universe appeared mutually inconsistent, and there was no consensus on the makeup of the energy density of the universe. The ΛCDM model can be extended by adding cosmological inflation, quintessence, and other elements that are current areas of speculation and research in cosmology.
The model includes **a single originating event, the "Big Bang", a singularity,** which was not an explosion, but the abrupt appearance of expanding spacetime containing **radiation at temperatures of around 10^{15}** K. This was immediately (within 10^{-29} seconds) followed by an **exponential expansion of space by a scale multiplier of 10^{27}** or more, known as cosmic inflation. The early universe remained hot (above **10,000 K) for several hundred thousand years**, a state that is **detectable** as a residual cosmic microwave background, or **CMB**, a very low energy radiation emanating uniformly from all parts of the sky.

IIB. The Hypothesized Thermal History of the Universe

We will briefly summarize the thermal history of the universe, from the Planck era to the present. As we go back in time, the universe becomes hotter and hotter and thus the amount of energy available for particle interactions increases. As a consequence, the nature of interactions goes from those described at low energy by long range gravitational and electromagnetic physics, to atomic physics, nuclear physics, all the way to high energy physics at the electroweak scale, grand unification (perhaps), and finally quantum gravity. The last two are still uncertain since we do not have any experimental evidence for those ultra high energy phenomena, and perhaps Nature has followed a different path.

In principle, one can trace the evolution of the universe from its origin till today. According to the best accepted view, the universe must have originated at the Planck era (10^{19} GeV, 10^{-43} s) from a quantum gravity fluctuation. Needless to say, we don't have any experimental evidence for such a statement: Quantum gravity phenomena are still in the realm of physical speculation. However, it is plausible that a primordial era of cosmological inflation originated then. Its consequences will be discussed below. Soon after, the universe may have reached the Grand Unified Theories (GUT) era (10^{16} GeV, 10^{-35} s). Quantum fluctuations of the inflaton field most probably left their imprint then as tiny perturbations in an otherwise very homogenous patch of the universe. At the end of inflation, the huge energy density of the inflaton field was converted into particles, which thermalized and became the origin of the hot Big Bang as we know it. Such a process is called reheating of the universe. Since then, the universe became radiation dominated. It is "probable" (although by no means certain) that the asymmetry between matter and antimatter originated at the same time as the rest of the energy of the universe, from the decay of the inflaton. This process is known under the name of baryogenesis since baryons (mostly quarks at that time) must have originated then, from the leftovers of their annihilation with antibaryons.

IIC. List of Challenges with the ΛCDM Big Bang Theory (BBT)

The Big Bang Theory **makes no testable predictions**. It is derived by fitting six parameters to minimize errors.
Methodology of BBT: BBT has no predictive power. **It's origin is using six parameters to curve fit the model** to known measurements. When faced with discrepancies between theory and observation, cosmologists habitually react by adjusting or adding these parameters to fit observations, propose additional hypotheses, or even propose "new physics" and ad hoc solutions that preserve the core assumptions of the existing model.
The BBT is based on the unverified core assumptions of the Cosmological Principle, namely that,

- ♦ The universe is isotropic and homogenous space at sufficiently large scales > 100 Mpc.
- ♦ However, The Cosmological Principle is manifested false within the distance scales that can be verified.

BBT Violates the Second Law of Thermodynamics: How did the universe start with such a Low Entropy?
The unknown nature and existence of Cold Dark Matter. The unknown nature and existence of Dark energy
Without the above sources of matter, the universe would be younger than the oldest stars, which is a contradiction.
Value of Cosmological Constant is one of the hugest inconsistencies in Physics. Off by 120 orders of magnitude!
Inflation Theory that requires initial conditions so unlikely that the probability that it happened purely by chance is greater than the probability of expansion by the Theory of Inflation.
Inflation requires a density 20 times higher than that implied by nucleosynthesis.
Postulates that the universe springs from a singularity. A singularity is a point of infinite density, infinite pressure, infinite temperature, and zero volume. At best, an extremely unstable state that is beyond the known laws of physics.
There is no known science that covers this, that is, no known physical laws. At best it is veiled by the Planck era.
A singularity is a thermodynamic dead end. Cannot return to other states.
None of Laws/Forces of Nature apply to Inflation, including GR. No event horizon around it. No spatial direction.
Friedmann Model breaks down at a singularity. No shell in which to define density. There is no space to put matter.
String Theory (M-Theory): Particles consist of one dimensional or two dimensional (called "branes") entities.
Absence of magnetic monopoles.
Assumption is that the only force on a cosmological scale is gravity. The force of gravity is 10^{-39} times smaller than E-M, but huge magnetic fields in space and indication of huge voltages and charge differences.
There is no explanation for the absence of anti-matter.
Expansion from a Singularity cannot produce rotational momentum required for galaxies and planets.
Confined gas molecules will produce a turbulence, destructive to a flat universe.

Latest Conflict with Big Bang Theory - Latest Discoveries from the James Webb Telescope

The James Webb telescope, looking back to 400,000 years after the Big Bang, has discovered at least five massive galaxies. This is inconsistent with the Big Bang Model. These massive galaxies would have to grow 20 times faster than the Milky Way. For these young galaxies, the BBT predicts galaxies 10 to 100 times smaller.

The Tenuous Link of the Stellar Distance Ladder

One of the Core Principles of the Current Big Bang Theory (BBT) of the Universe is the Validity of the use of Stellar Distance Ladders to measure the distance to galaxies. However, less than 1% of the visible universe has a Distance Ladder that is verifiable by direct measurement.

Inconsistencies and Challenges - Cosmological "Tensions"

Differences in measured values of Hubble Constant from Redshift vs. Recession Velocity and CMBR Uniformity
High redshift galaxy observations predict a higher star formation efficiency then BBT Planck CMB.

"Population of surprisingly massive galaxy candidates with stellar masses of order of 10^9 x Mass of the Sun, $M_\odot$.

See this Review Article for an Up-to-date Summary of the Challenges and "Tensions" facing the BBT:

Challenges for ΛCDM: An update, L. Perivolaropoulos and F. Skara, arXiv:2105.05208v3 6 Apr 2022

Successes of the BBT

The ΛCDM model has been remarkably successful in explaining most properties of a wide range of cosmological observations including the accelerating expansion of the Universe (Perlmutter et al. 1999; Riess et al. 1998), the power spectrum and statistical properties of the cosmic microwave background (CMB) anisotropies (Page et al. 2003), the spectrum and statistical properties of large scale structures of the Universe.

III. Some Key Historical Events and Investigative Methods

Key Mathematical Concepts for Correct Modeling of Planetary Motion

1605: Kepler's Laws
Example: Applications: Elliptical Model for Planet's Orbits. Lunar Conic Model, Time of Flight, Polar Model

1686: Newton's Laws (supercede Kepler's Laws.):
Example Applications: Calculating the Trajectory of rocket to Moon and back. 3 Body and 4 Body (Sun, Earth, Moon, and Rocket.)

Key Concepts and Discoveries of Cosmology

1. Mathematical Basis of Big Bang Cosmology: Einstein's General Theory of Relativity
In 1917 Einstein developed his General Theory of Relativity (GR).

2. In 1922 Friedmann developed a solution of GR that showed that the universe is not static, but predicted that the universe will expand. In 1927 Lemaitre came up with a model that included mass density and pressure. He showed a linear relationship between expansion of the universe and distance. This relationship was later discovered by Hubble. Hubble's Law. Based on the idea of running cosmic clock backwards, he proposed the Big Bang Model.

3. In 1929, measurements of the distance and the velocity of how fast galaxies are moving away from us were made by Edwin Hubble. The relationship he discovered between distance and velocity is know as Hubble's Law.

4. In 1948 a prediction of the existence of Cosmic Microwave Background Radiation (CMBR) was made by George Gamow.

5. In 1950's it was thought that the light elements, such as hydrogen and helium. were formed in stars. However, the observed % of helium was too high to be formed from the interior temperatures of stars. The percentage of Helium can be explained by the BBT, i.e, the universe was so hot that it could produce a high percentage of helium.

6. In 1964 Penzias and Wilson, while calibrating a radio telescope accidentally discovered this (CMBR). Based on GR, the discovery of CMBR, and Hubble's Law the Big Bang Theory was proposed. To verify that the CMBR originated from a BB, in 1989 the COBE spacecraft was launched to determine if the temperature variations of the CMBR were consistent with a Big Bang Origin. The uniformity of CMBR agreed with BBT.

7. Observations of rotational velocity of galaxies implied the existence of a new form of matter: Cold Dark Matter.

8. 1960's: The Development of the ΛCDM (Lambda Cold Dark Matter) Model

9. In the 1980s the concept of inflation was proposed to explain the fine tuning of the universe. Cosmic inflation, cosmological inflation, or just inflation, is a theory of exponential expansion of space in the early universe. The inflationary epoch is believed to have lasted from 10^{-36} seconds to between 10^{-33} and 10^{-32} seconds after the Big Bang. It requires a fine tuning of one part in 10^{50}. XIX discusses the serious problems with the validity of this theory.

10. In 1998, it was observed that the rate of expansion of the universe increased. This increase was attributed to a new form of energy called dark energy. In 2022, it was found to increase 5% to 9% even faster than thought.
The Greek letter Λ (lambda) is used to represent the cosmological constant, which is currently associated with a vacuum energy or dark energy in empty space that is used to explain the contemporary accelerating expansion of space against the attractive effects of gravity. A cosmological constant has negative pressure,

IV. List of Some of the Cosmological Topics To Be Covered

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4. Cosmological Distances: The Horizon Problem. Proper and Comoving Distance
5. Newtonian Energy Derivation of the Rate of Expansion, H
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Some Cosmology Nomenclature

$a(t)$	Scale factor of the Universe
$a_{\ell m}$	Multipole of $\Delta T/T$
C_ℓ	Spectrum $\langle \lvert a_{\ell m}\rvert^2\rangle$ of CMB anisotropy
f	Occupation number
$g_{\mu\nu}$ (g_{ij})	Spacetime (space) metric tensor
h_{ij}	Gravitational wave amplitude
H (H_0)	Hubble parameter $\dot{a}/a$ (present value)
k $(\mathbf{k})$	Comoving wavenumber (wave vector)
L $(\mathcal{L})$	Lagrangian (Lagrangian density)
n	Number density
n_s	Spectral index of ζ
N	Hubble times of observable inflation
P	Pressure
$\mathbf{p}$ (p) (p^μ)	Momentum (magnitude of) (4-momentum)
$\mathcal{P}_g$	Spectrum of a perturbation g
r	Tensor fraction $\mathcal{P}_h/\mathcal{P}_\zeta$
$T^{\mu\nu}$	Energy momentum tensor
$\mathbf{v}$	Fluid velocity
V	Fluid velocity scalar
$V(\phi)$	Scalar field potential
x (x^μ) (x^i)	Comoving distance (spacetime coordinates) (space coordinates)
w	Ratio P/ρ for a fluid
z	Redshift
δ	Density contrast $\delta\rho/\rho$
ϵ	Slow-roll flatness parameter $\frac{1}{2}M_\mathrm{P}^2(V'/V)^2$
ζ	Primordial curvature perturbation
η	Slow-roll flatness parameter $M_\mathrm{P}^2 V''/V$
η	Conformal time $d\eta = dt/a$
$\eta_{\mu\nu}$	Metric tensor (Minkowski coordinates)
ρ (ρ_0)	Energy density (of present Universe)
Π	Anisotropic stress scalar
ϕ	Scalar field
Φ	Newtonian peculiar gravitational potential
Φ, Ψ	Metric perturbations
φ	Conformal inflaton field perturbation $a\delta\phi$
Ω_s	Present ρ_s/ρ of species 's"
R_c	Radius Hubble Sphere (Region where galaxies recede subliminally)
$g_{\mu\nu}$	The Metric $g_{\mu\nu}$. A rank two symmetric tensor that encodes information about geometry.
$T_{\mu\nu}$	Einstein Stress-energy Tensor which describes matter and energy distributions.
$R^\delta{}_{\mu\nu}$	Riemann Tensor is a math construct used to characterise the curvature of space-time.
$R_{\mu\nu}$	The Ricci Tensor is a contraction of the Riemann Tensor.
R	The Ricci Scalar is a contraction of the Ricci Tensor.
$G_{\mu\nu}$	The Einstein tensor $G\mu\nu$ is defined in terms of the Ricci tensor and scalar.

V. Mathematical Basis of Big Bang Cosmology: Einstein's General Relativity and FLRW

In 1915, **Einstein** developed his **General Theory of Relativity (GR)**. GR consists of a number of field equations that relate the geometry of spacetime to the distribution of matter within it. GR provides a deep physical and geometrical description of how mass/energy determines the dynamics of the universe.
The evolution space-time of the universe is guided by the Einstein Field Equation.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{M_{\rm pl}^2} T_{\mu\nu}$$

where the spacetime metric $g_{\mu\nu}$ and its corresponding Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are related to energy content expressed through the Einstein Symmetric, order-2, Energy-Momentum Tensor $\boldsymbol{T_{\mu\nu}}$. Briefly, the Einstein equations equate the matter that's present in a spacetime with the spacetime's geometry.

In 1917, Schwarzschild solved the Einstein equations under the assumption of spherical symmetry, two years after their publication. The most obvious spherically symmetric problem is that of empty space outside a planet or star. The mass curves space-time and thus affects the particles moving nearby. The space-time interval in spherical coordinates in the Schwarzschild solution is.

$$ds^2 = (1 - \frac{2GM}{c^2 r})c^2\, dt^2 - \frac{dr^2}{(1 - \frac{2GM}{c^2 r})} - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\psi^2$$

Schwarzschild Equation Prediction of the Formation of a Black Hole

The definition of proper time, τ (tau), is the time interval for an observer at rest. In Minkowski space time $ds^2 = dt^2 - dx^2$, for $dx = 0$, $dt = d\tau$. Similarly in the Schwarzschild Metric if we have an observer at rest then dr = 0...etc and then proper time should be $dt = d\tau$ (like in the SR case giving ds = (1−2M/r)dt = (1−2M/r)$d\tau$

the first term T_{00} in the Einstein $T_{\mu\nu}$ Tensor is T_{tt}. If you take the distance r to be equal to $r_s = 2\frac{GM}{c^2}$

then the time factor, T_{tt}, which is equal to 1 - 2GM/c^{2}r in ds^2 becomes 0, so the value of ds^2 is undefined.
It becomes a singularity. This value of the radius = r_s is called the **Schwarzschild radius**.

From the Schwarzschild Metric**,** if we plot the passage of time, $\Delta\tau$, versus the distance to the center, r, the relation is:

$$\Delta\tau(r) := 1 - 2\frac{GM}{r}$$

The Plot below show that at the Event Horizon that the passage of proper time, τ, slows to 0, that is, time stops.

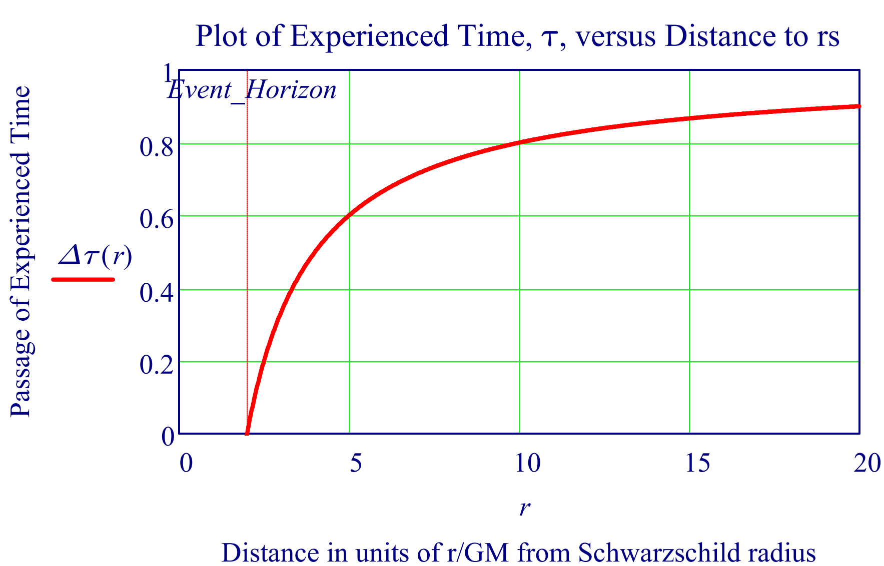


Image of the Event Horizon Captured by the Event HorizonTelescope

In 1922, **Friedmann–Lemaître–Robertson–Walker (FLRW)** proposed a Relativistic Space-Time Metric that is the basis for an exact solution of Einstein's field equations of General Relativity; it is based on the assumption of a homogeneous, isotropic, and expanding (or otherwise, contracting) universe. The general form of the metric follows from the assumption of homogeneity and isotropy of space in the universe; Under these set of assumptions, Einstein's field equations are only needed to derive the scale factor of the universe as a function of time.

If we model the universe as a homogeneous, isotropic with spherical coordinates, we obtain the the Friedmann metric. By defining a cosmic scale factor, "***a(t)***", which is a function of time. This scale factor parametrizes the expansion of space. The radius, r, is transformed to a comoving coordinate. Furthermore, the radius of curvature is also affected by cosmic expansion so it can be expressed in terms of the scale factor and a constant k

The Friedmann–Lemaître–Robertson–Walker (FLRW) Relativistic Space-Time Metric in terms of "a" is:

$$ds^2 = -c^2dt^2 + a^2(t)[\frac{dx^2}{1-\kappa\frac{x^2}{R^2}} + x^2 d\Omega^2] \qquad \text{where} \qquad Kt = \frac{k}{a^2 t}$$

Note: **"a"** is NOT the acceleration, it is the Scale Factor.

Based on this metric and its solution of the Einstein's Field Equations give the two Friedmann Equations. The first is:

$$\frac{\dot{a}^2 + kc^2}{a^2} = \frac{8\pi G\rho + \Lambda c^2}{3} \qquad G := 6.6743\cdot 10^{-11}\cdot m^3\cdot kg^{-1}\cdot s^{-2}$$

which is derived from the 00 component of Einstein's field equations. The second is:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}$$

This is the evolution equation for the scale factor, a.

where **a is the scale factor**, G, Λ, and c are universal constants. G is Newton's gravitational constant, Λ is the cosmological constant with dimension length^{-2}, and c is the speed of light in vacuum. ρ and P are the volumetric mass density and the pressure, respectively. k is constant throughout a particular solution, but may vary from one solution to another. The symbol "a" is defined as the scale factor which changes with time, ρ and p are the volumetric mass density and pressure. They may vary from one solution to another. The expansion of the universe (**ȧ/a**) can be measured**.**

In the Friedmann model, **H ≡ ȧ/a and is defined as the Hubble parameter, which evolves with time.**

In order to solve the Friedmann Equation, we need to define the behavior of the mass/energy densityρ(a) of any given mass/energy component. Recall the basic GR paradigm:

Density Determines the Expansion
Expansion Changes the Density

Density Components: Each component will lead to a different evolution in redshift and a different Model

Matter $\rho_m(t) = \rho_{m0}\cdot a^{-3}(t)$ $\qquad \rho_{rad}(t) = \rho_{rad0}\cdot a^{-4}(t)$ $\qquad \rho_\Lambda(t) = \rho_\Lambda = constant$

$$\rho_0 = \frac{3\cdot H_0^2}{8\cdot\pi\cdot G} \qquad H^2 = \frac{8\pi\cdot G\cdot\rho + \Lambda\cdot c^2}{3} - \frac{k\cdot c^2}{a^2}$$

Seconds in a Billion (Giga) Years, Gyr

$$Gyr := 3600\cdot 24\cdot 365.24\cdot 10^9\cdot s$$

$$Mpc := 3\cdot 10^{19} km \qquad H_0 := 73\frac{km}{s}\cdot(Mpc)^{-1} \qquad \Omega_M = \frac{8\pi G\cdot\rho}{3\cdot H_0^2} \qquad \Omega_\Lambda = \frac{\Lambda\cdot c^2}{3\cdot H_0^2}$$

When a_0 = 1

$$H_0^2\cdot\Omega_\Lambda = \Lambda\cdot c^2 3$$

The Equation of State for a Simple Fluid

• Usually written as $P = w\,\rho$ P is the Pressure and ρ is the density.
• This is not necessarily the best way to describe the matter/energy density; it implies a fluid of some kind…
This may be OK for the matter and radiation we know,
but maybe it is not an optimal description for the dark energy
• Special values:
 $w = 0$ means $P = 0$, e.g., non-relativistic matter
 $w = 1/3$ is radiation or relativistic matter
 $w = -1$ looks just like a cosmological constant
… but it can have in principle any value, and it can be changing in redshift

Density Determines the Expansion
Expansion Changes the Density

Evolution of the Density, ρ

Continuity Equation
(Specifies that matter is conserved.)

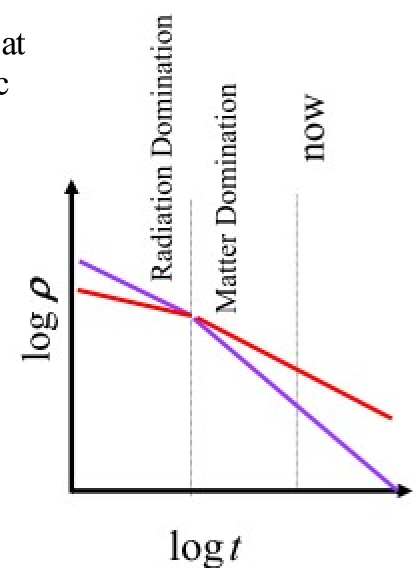
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \boldsymbol{v}) = 0$$

Generally, $\rho \sim a^{-3(w+1)}$

• Matter dominated ($w = 0$): $\rho_m \sim a^{-3}$ Contінity Equation: $\rho \sim a^{-3}$
• Radiation dominated ($w = 1/3$): $\rho_r \sim a^{-4}$ Wavelength stretched with z
• Cosmological constant ($w \sim -1$): ρ_Λ = constant Constant Vacuum Energy
• Dark energy with ($w < -1$) e.g., $w = -2$: $\rho_{dm} \sim a^{+3}$
 – Energy density increases as is stretched out!
 – Eventually would dominate over even the energies holding atoms together! ("Big Rip")

In a mixed universe, different components will dominate the global dynamics at different times Note that in principle, w could be a function of time, density, etc

• Radiation density decreases the fastest with time
 – Must increase fastest on going back in time
 – Radiation must dominate early in the Universe
• Dark energy with $w \sim -1$ dominates last; it is the dominant component now, and in the future



Models With Both Matter & Radiation ==>

Harder to solve for $\rho(t)$

However, to good approximation, we can assume that $K = 0$ and either **radiation** or **matter** dominate

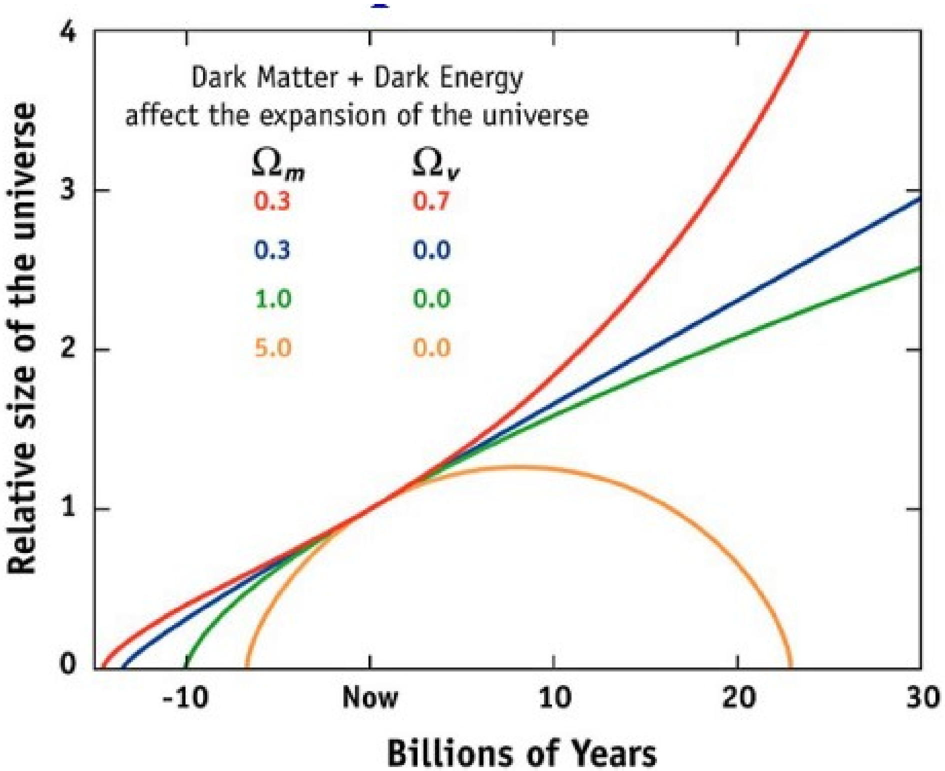
		γ-dom	m-dom
"α" is the symbol for *proportional to*	a(t)	α $t^{1/2}$	α $t^{2/3}$
• Matter (m) dominated ($w = 0$):	ρ_m α a^{-3}	α $t^{-3/2}$	α t^{-2}
• Radiation (γ) dominated ($w = 1/3$):	ρ_γ α a^{-4}	α t^{-2}	α $t^{-8/3}$

• Cosmological Constant (Λ) ($w = -1$): $\Lambda \sim a^{-\lambda t}$

Models in Cosmology

Generally: $\frac{8\pi G\rho}{3} = H_0^{\,2}\left(\Omega_{\Lambda,0} + \Omega_{m,0}a^{-3} + \Omega_{\gamma,0}a^{-4}\right)$

Example of Models



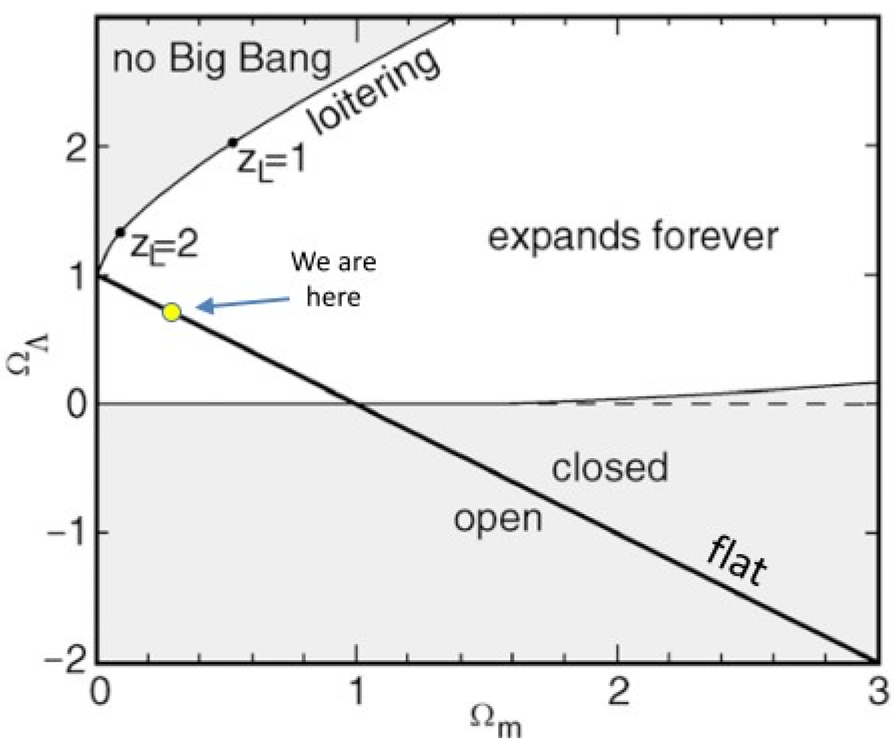
Consider Several Simple Models

Refer to Section VIII

- $k = 0$, matter dominated, Einstein de Sitter
- $k = 0$, radiation dominated
- $k < 0$, $\rho = 0$, Milne Model
- $k < 0$, $\rho > 0$
- $k > 0$
- Λ dominated

k is the curvature of space

Classification of Models



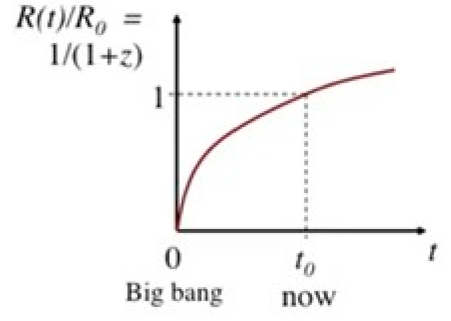
(Ignoring Ω_{rad}, since it is negligible for most of the history of the universe)

Distances in Cosmology

So far, we've found out how to compute different cosmological models. But what good are they?

The basic goal of cosmology is to figure out in what model universe do we live.
Models are basically distinguished by their history of the expansion rate,
how their scale factor changes as a function of time.

If we can figure out which curve of those we live on, we know we'll know about cosmological parameters.



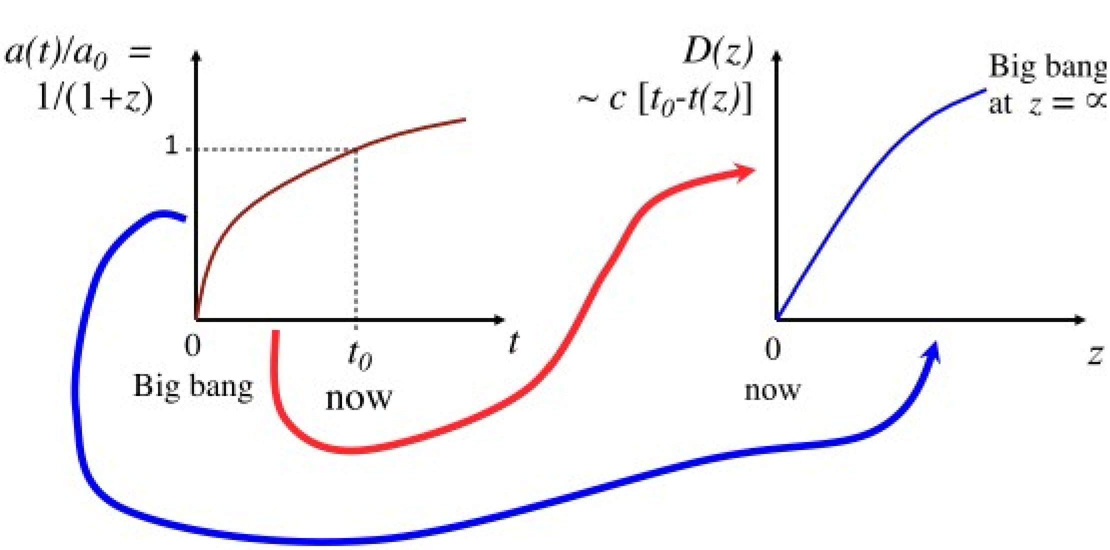
The expansion factor R(t) is simply related to redshift, z, that is an observable quantity, and that's an easy part. The other axis is the time axis. Now unfortunately, this them galaxies do not carry gigantic clocks on them.

So it's very hard to figure out what is the look back time between us in some distant point, in a way that can be measured. So instead of that, what we do is we do **we transform coordinates,**
instead of the look back time, we can use distance which is simply time multiplied by the speed of light.
Distances in principle we can measure so we flipped the star Game and instead of expansion factor R(t), we use the redshift, which is an observable quantity. And instead of the time we use a distance, which we can figure out how to measure in some way.

So essentially, all cosmological tests boil down to this
We have to s**omehow measure a set of distances to a points as a function of redshift.** And because the whole thing just scales with Hubble constant, we only need to determine the shape of that curve.

So let's figure out how to measure distances in cosmology. A convenient unit of distance is Hubble distance, which is simply speed of the light divided by the Hubble constant. The Hubble constant has dimensions of one over time.

The Basis of Cosmological Tests



All Cosmological Tests
essentially consist of
comparing some measure
of (relative) distance,
$D(z)=c^*(t_0-t_z)$
(or look-back time)
to redshift, z.
Absolute distance scaling
is given by the H_0.

Cosmological Tests: The Why and How

- Model equations are integrated, and compared with the observations
- The goal is to determine the global geometry and the dynamics of the universe, and its ultimate fate
- The basic method is to somehow map the history of the expansion, and compare it with model predictions
- A model (or a family of models) is assumed, e.g., the Friedmann-Lemaitre models, typically defined by a set of parameters, e.g., H_0, $\Omega_{0,m}$, $\Omega_{0,\lambda}$, q_0, etc.
- Model equations are integrated, and compared with the observations

Distances in Cosmology

A convenient unit is the Hubble distance or radius, $D_H = c\ H_0 = 4.283\ h_{70}^{-1}$ Gpc $= 1.322\ 10^{28}\ h_{70}^{-1}$ cm

and the corresponding Hubble time, $t_H = 1/H_0 = 13.98\ h_{70}^{-1}$ Gyr $= 4.409\ 10^{17}\ h_{70}^{-1}$ s $= 13.02$ Gyr

At low z's, distance $D \approx z\ D_H$. But more generally, the comoving distance, D_C to a redshift z is:

$$D_C = D_H\left(\int_0^z \frac{1}{E(z)}\,dz\right)$$

In general, this integral is not solvable analytically

$$E(Z) = \sqrt{\Omega_k\cdot(1+z)^2 + \Omega_{0m}\cdot(1+z)^3 + \Omega_{0r}\cdot(1+z)^4 + \Omega_{0\Lambda}}$$

Note: All Distances and Time scale linearly with the Nubble Constant, H

$$\Omega_k = 1 - \Omega_r - \Omega_m - \Omega_\Lambda$$

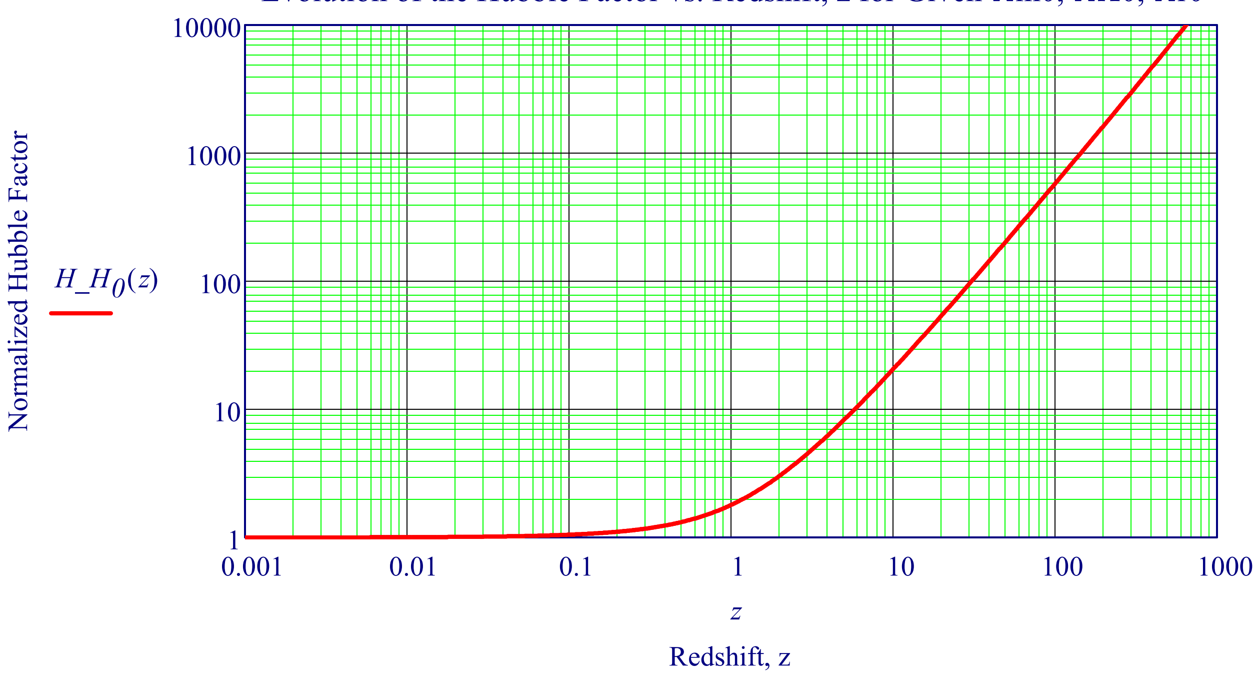
Λ-CDM Model Parameters

$$\Omega_{r0} := 8.7\cdot10^{-5} \qquad \Omega_{m0} := 0.317 \qquad \Omega_{\Lambda0} := 0.683$$

Dynamical Equation Specifying the Evolution of the Hubble Factor of Our Universe

$$\frac{H}{H_0} = H\_H_0(z) := \sqrt{\Omega_{m0}\cdot(1+z)^3 + \Omega_{r0}\cdot(1+z)^4 + \Omega_{\Lambda0}}$$

Evolution of the Hubble Factor vs. Redshift, z for Given Ωm0, ΩΛ0, Ωr0



Cosmological Distances: The Horizon Problem

There are fundamentally Two Kinds of Coordinates in a GR cosmology:

Comoving coordinates = Expand with the Universe.
Examples:
- Unbound systems, e.g., any two distant galaxies
- Wavelengths of massless quanta, e.g., photons
- Stretches relative to the Proper Coordinates

Proper coordinates = stay fixed, *space expands relative to them*.
Examples:
- Sizes of atoms, molecules, solid bodies
- Gravitationally bound systems, e.g., Solar system, stars, galaxies …

We introduce a **scale factor**, commonly denoted as **R(t) or a(t)**: a spatial distance between any two unaccelerated frames which move with their **comoving coordinates**
Computing a(t) and various derived quantities **defines the cosmological models**.
This is accomplished by solving the Friedmann Equation

1. Proper Distances

We define a proper distance, as the distance between two events, A and B, in a reference frame for which they **occur simultaneously** **($t_A = t_B$).**

$$(ds)^2 = (c\,dt)^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right]$$

and set $d\theta=d\varphi=0$ and $dt=0$, so that

$$s(t) = \int_0^s ds' = a(t)\int_0^r \frac{dr}{(1-kr^2)^{1/2}}$$

This has solutions:

$$s(t) = a(t)\cdot\begin{cases} \frac{1}{\sqrt{k}}\sin^{-1}(r\sqrt{k}) & \text{for } k>0 \\ r & \text{for } k=0 \\ \frac{1}{\sqrt{|k|}}\sinh^{-1}(r\sqrt{|k|}) & \text{for } k<0 \end{cases}$$

In a flat universe, **the proper distance** to an object is just its **coordinate distance,**

$$s(t) = a(t)\cdot r.$$

Because $\sin^{-1}(x) > x$ and $\sinh^{-1}(x) < x$,

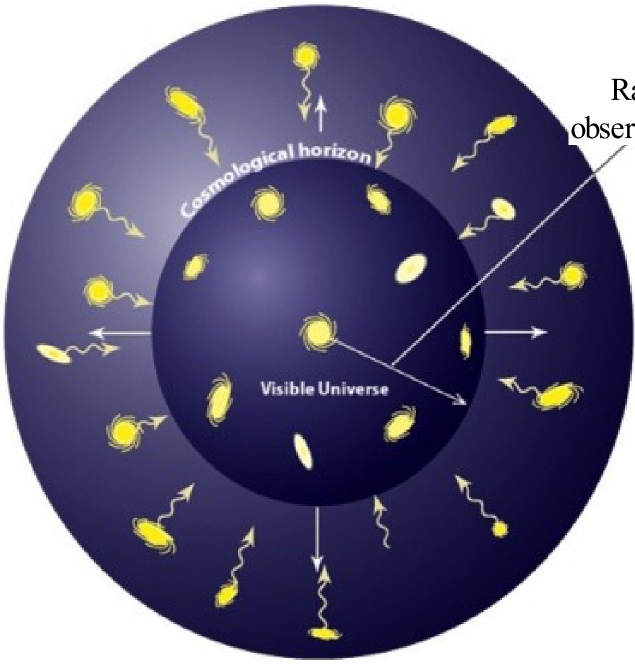
Universe Contracts (Closed) or Universe Expands (Open)

in a closed universe (k > 0)
the proper distance to an object is greater than its coordinate distance,
while in an open universe (k < 0)
the proper distance to an object is less than its coordinate distance.

The Horizon

The proper distance to the **furthest observable point** - the particle horizon - at time t is:

Horizon Distance: $s_h(t)$



Radius of the observable Universe

As the universe expands and ages, **an observer at any point is able to see increasingly distant objects** as the light from them has time to arrive. This means that, **as time progresses, increasingly larger regions of the universe come into causal contact with the observer.**

The proper distance to the furthest observable point, the particle horizon— at time t is the **horizon distance, $s_h(t)$.**

Again we return to the Robertson-Walker metric, placing an observer at the origin ($r = 0$) and let the particle horizon for this observer at time t be located at radial coordinate distance r_{hor}.

This means that a photon emitted at $t = 0$ at r_{hor} will reach our observer at the origin at time t.

Since photons move along null geodesics, $ds = 0$. Considering only radially traveling photons ($d\theta = d\varphi = 0$), we find

$$\int_0^t \frac{dt}{a(t)} = \frac{1}{c}\int_0^{r_{hor}} \frac{dr}{(1-kr^2)^{1/2}}$$

for $k = 1$ $\quad \sin\left(c\int_0^t \frac{dt}{a(t)}\right)$ $\qquad$ for $k = 0$ $\quad c\int_0^t \frac{dt}{a(t)}$ $\qquad$ for $k = -1$ $\quad \sinh\left(c\int_0^t \frac{dt}{a(t)}\right)$

If the scale factor evolves with time as $a(t) = t^{\alpha}$, we can see that the above time integral diverges as we approach $t = 0$, if $\alpha > 1$. **This would imply that the whole universe in is causal contact.**
However, α=1/2 and 2/3 in the radiation and matter-dominated regime, so there is a horizon.

The proper distance from the origin to r_{hor} is given by:

for $k = 0$

$$s_{hor}(t) = a(t)\int_0^{r_{hor}} \frac{dr}{(1-kr^2)^{1/2}} = a(t)\int_0^t \frac{cdt}{a(t)}$$

So $s_{hor}(t) = 2ct$ in the radiation-dominated era and $s_{hor}(t) = 3ct$ in the matter-dominated era. Notice that these distances are **larger than ct, the distance travelled by a photon in time t**. How could this be? The reason lies in our definition of proper distance, as the distance between two events measured in a frame of reference where those two events happen at the same time.
To understand this, consider a photon in emitted at comoving radial coordinate r_{hor} at time $t = 0$. We want to know what is the proper distance of that photon from ***our*** position, at $r = 0$, at a later time t. The coordinate of the photon at time t may be found by integrating

$$\int_0^t \frac{cdt}{a(t)} = \int_r^{r_{hor}} \frac{dr}{(1-kr^2)^{1/2}}$$

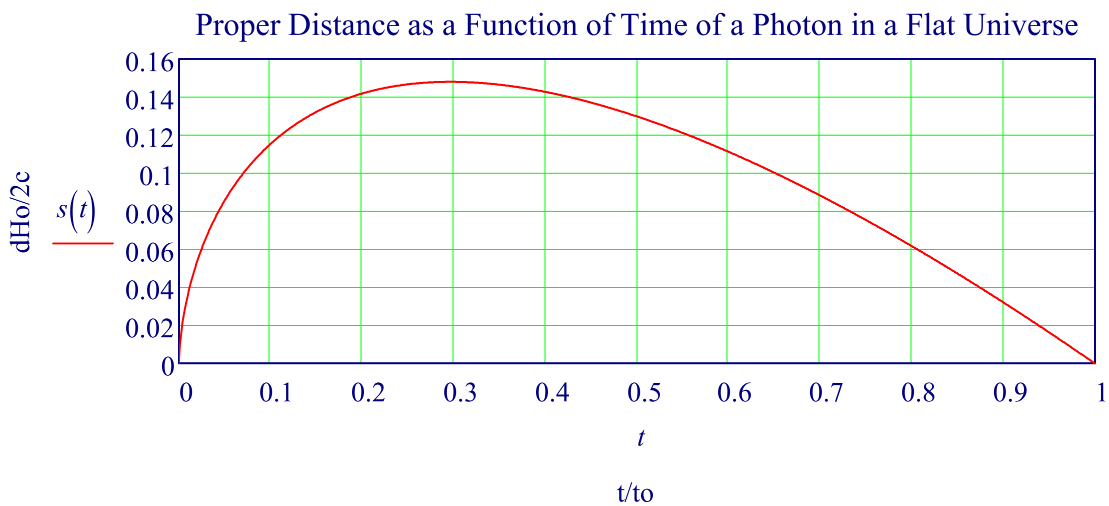
As before, we consider zero curvature models. Substituting for a(t) we obtain:

$$r = r_{\text{hor}} - \frac{2c}{H_0}\left(\frac{t}{t_0}\right)^{1/3}$$

where $\mathbf{t_0 = 2t_H/3}$ **is the present age of the universe**. Recalling that $\mathbf{r_{hor} = 2c/H_0}$, and that **the proper distance** in a flat universe is just **s(t) = a(t) · r**, we find that the proper distance of the photon from Earth as a function of time is

$$s(t) := \frac{2c}{H_{0.}} \cdot \left[\left(\frac{t}{t_{0.}}\right)^{\frac{2}{3}} - \frac{t}{t_{0.}}\right] \qquad \text{for } k = 0$$

Proper distance as a function of tinie of a photon emitted from the present particle horizon at the time of the Big Bang. The proper distance is expressed as. function of $2c/H_0$, the present horizon distance in a flat universe.



We can now see that **the initial expansion actually carried the photon away from Earth.** Although the photon's co-moving coordinate was always decreasing from an initial value r_{hor} towards Earth's position at r = 0, the scale factor a(t) increased so rapidly that **<u>at first the proper distance between the photon and Earth increased with time</u>**.

Expansion and the Hubble's Law

Consider a point at a **comoving distance x**. At some time t it will be at a radial distance r(t) = a(t) x , where a(t) is the expansion factor. We will designate values for "here and now" with a subscript 0, t_0 = now, and $a_0 = a(t_0) = 1$. The recession velocity is:

$$\boldsymbol{v}(\boldsymbol{r},t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}(t) = \frac{\mathrm{d}a}{\mathrm{d}t}\boldsymbol{x} \equiv \dot{a}\,\boldsymbol{x} = \frac{\dot{a}}{a}\boldsymbol{r} \equiv H(t)\,\boldsymbol{r}$$

Where $H(t) := \dfrac{\dot{a}}{a}$ is the normalized expansion rate

$$\Delta\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}+\Delta\boldsymbol{r},t) - \boldsymbol{v}(\boldsymbol{r},t) = H(t)\,\Delta\boldsymbol{r}$$

Which is the same as the Hubble's law: $v = H_0\,D$

H_0 is the value of the **expansion rate here and now**.
Note that it is not a constant, but it depends on a(t).

VI. Newtonian Energy Derivation of the Rate of Expansion, H

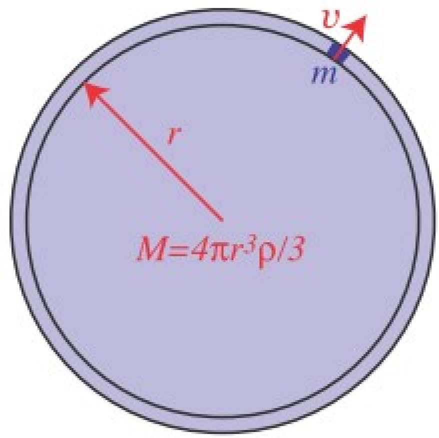
Consider a test particle of mass m as part of an expanding spherical shell of radius r & total mass M.

$$r(t) = a(t)\cdot x \qquad x = \frac{r(t)}{a(t)} \qquad\qquad v(r,t) = \frac{d}{dt}r(t) = \frac{da}{dt}x = \frac{da}{dt}\cdot\frac{r}{a} = \frac{a}{a}\cdot\frac{\dot{a}}{a} = H(t)\cdot r$$

Note: **"a"** is NOT the acceleration, it is the Scale Factor.

By Conservation of Energy, E = Constant

$$Energy = \frac{1}{2}m\cdot v^2 - \frac{GMm}{r} = Constant$$



$$\frac{1}{2}\left(\frac{d}{dt}r\right)^2 - \frac{GM}{r} - Constant = \blacksquare$$

$$\mathrm{M} = \frac{4}{3}\pi r^3\cdot\rho \qquad r_{BA} = a(t)\cdot\frac{x}{BA}$$

$$\frac{1}{2}\left(\frac{1}{r}\cdot\frac{d}{dt}r\right)^2 - \frac{G\cdot M}{r^3} - \frac{Constant}{r^2}$$

$$\boxed{H^2 = \frac{8\pi\cdot G\cdot\rho}{3} - \frac{2\cdot Energy}{a^2}} \qquad H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{K}{a^2}$$

The Two Friedmann Equations can be reduced to:

$$H^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \qquad \rho_\Lambda \equiv \frac{\Lambda}{8\pi G_N} \qquad \rho_{\mathrm{tot}} \equiv \rho + \rho_\Lambda$$

Expansion = Density - Curvature **ρ_Λ = Cosmological Constant Energy Density** **ρ_{tot} = Total Energy Density**

For a given value of H, there is a special value of the density which would be required in order to make the geometry of the Universe flat, that is, $k=0$. This is known as the critical density ρ_c

$$\rho_c \equiv \frac{3H^2}{8\pi G_N} \qquad \rho_c := 9.9\cdot 10^{-30}\cdot\frac{gm}{cm^3}$$

Sources of Matter and Energy

In General Relativity, all of the sources of matter and energy are included and contribute to the total energy density, ρ_{tot}. **The energy density today of each component is Normalized to the Critical Density, ρ_c,** (See below: $\Omega_{component}$) that is used in the definition of the corresponding "Omega parameter", $\mathbf{\Omega}$.

$$\Omega_{component} = \frac{\rho_{component}}{\rho_{c\_z0}} \qquad \text{Thus we have: } \mathbf{\Omega} = \Omega_{baryon} + \Omega_{cdm} + \Omega_{radiation} + \Omega_{DE}$$

Here Ω_{baryon} is the baryon content, Ω_{cdm} is the amount of cold dark matter, $\Omega_{radiation}$ is the radiation content, and Ω_{DE} is the contribution from dark energy. If $\Omega = 1$ that means the density is equal to the critical density, ρ_c, at $z=0$, so we have a flat Universe ($k=0$).

VII. Equations for Cosmological Parameters: Hubble & Scale Factors, z, Ωs, Density, Temp, V

Definitions and Equations below came from: *Introduction to Cosmology*, by Barbara Ryden

Plots of these Cosmic Parameters are on the Following Pages

Define Constants

Seconds per Billion (Giga) Years

$c := 3\cdot 10^{8}\cdot\frac{m}{s}$ $\quad G := 6.67\cdot 10^{-11}\frac{m^3}{kg\cdot s^2}$ $\quad H_0 := \frac{1}{4.55\cdot 10^{17}}\cdot s^{-1}$ $\quad Gyr := 3600\cdot 24\cdot 365.24\cdot 10^{9}\cdot s$

Create an Exponential Time and Scale Factor, OM, Spanning 26 Orders of Magnitude:

Densities of our Universe

$OM := 26 \quad i := 0..100\cdot OM + 400 \quad a_i := 10^{0.01\cdot i - OM}$

DEFINE: Density Ω, H, da dt, Proper time, t, Diameter, Velocity, Mass Ratios, H(z)

$\rho_0 = \frac{3\cdot H_0{}^2}{8\cdot\pi\cdot G}$ $\quad \rho_0 := 8.6443584621592\cdot 10^{-27}\cdot\frac{kg}{m^3}$

In flat universe total density ρ = critical density ρ_0

Normalized radiation energy density for photons + neutrinos

$\Omega_{r0} := \frac{4.005\cdot 10^{-14}}{\rho_0\cdot c^2}\cdot(1+0.69)\cdot\frac{J}{m^3}$ $\quad \Omega_{r0} = 8.7\times 10^{-5}$

Redshift

$a(t) = \frac{1}{1+z}$ $\quad z = \frac{1}{a} - 1$

$\Omega_{m0} := 0.268 + 0.049$ **Dark matter + baryonic matter**

Hubble Parameters

$\Omega_{\Lambda 0} := 1 - \Omega_{r0} - \Omega_{m0}$ **Dark energy for flat universe**

$$H_i := H_0\cdot\sqrt{\frac{\Omega_{r0}}{(a_i)^4} + \frac{\Omega_{m0}}{(a_i)^3} + \Omega_{\Lambda 0}}$$

$\Omega_{\Lambda 0} = 0.683$

Friedmann equation for a flat universe after inflation ends and radiation epoch begins

$$da\_dt(a) := H_0\cdot\sqrt{\frac{\Omega_{r0}}{a^2} + \frac{\Omega_{m0}}{a} + \Omega_{\Lambda 0}\cdot a^2}$$

Calculate the Cosmic Proper Time (t) and Lookback time (tL). Inflation Epoch Ends at 10^- 33 seconds

Distance to a galaxy is defined as the proper distance $d_p(t)$. The length of time light has traveled t_0 - t_e is lookback time, t_L.

$$t_i := \int_0^{a_i}\frac{1}{da\_dt(a)}\,da \qquad t_{L_i} := \int_0^{z_i}\frac{1}{(1+z\varepsilon)\cdot H(z\varepsilon)}\,dz\varepsilon$$

$\frac{t_{100\cdot OM}}{Gyr} = 13.682$ $\quad \frac{t_{3000}}{Gyr} = 173.216$

$Now := t_{100\cdot OM}\cdot s^{-1}$

Calculate the Diameter in Meters of Observable Universe Dou = 2*comoving distance

$dD(a) := \frac{2\cdot c}{a\cdot da\_dt(a)}$ $\quad Initial := 1\cdot 10^{-100}$ $\quad da\_dt_i := da\_dt(a_i)$

$Dou_i := Integral\_dD(Initial, a_i, 500)$ $\quad Dou_0 = 292.689\,m$ $\quad \frac{Dou_{100\cdot OM}}{c\cdot Gyr\cdot s} = 92.572\,\frac{1}{s}$

D = Scaled Up Diameter of Universe that was formerly observable at 10^{-33} second

$D_i := \frac{a_i}{a_0}\cdot Dou_0$ $\quad D_0 = 292.689\,m$ $\quad \frac{D_{100\cdot OM}}{Dou_{100\cdot OM}} = 33.397$

Calculate Recessional Velocities

Temperature (K)

$vrou_i := H_i\frac{Dou_i}{2}$ $\quad Vou := \frac{4\pi}{3}\cdot\left(\frac{Dou}{2}\right)^3$ $\quad V_i := \left(\frac{a_i}{a_0}\right)^3\cdot Vou_0$ $\quad vr_i := H_i\cdot\frac{D_i}{2}$ $\quad T_i := \frac{2.725}{a_i}$

Mass Densities: Radiation, Mass, Λ, and Total

$$\rho r_i := \frac{\Omega_{r0}}{(a_i)^3}\rho_0 \qquad \rho m_i := \frac{\Omega_{m0}}{(a_i)^3}\rho_0 \qquad \rho\Lambda_i := \Omega_{\Lambda 0}\rho_0 \qquad \rho := \rho r + \rho m + \rho\Lambda$$

$$\rho_0 = 2.741 \times 10^{51}\,\frac{kg}{m^3}$$

Mass (Mou) and Energy (Eou) of Dark and Baryonic Matter and Energy in Observable Universe

$$Mou_i := \rho m_i \cdot Vou_i \qquad Erou_i := \rho r_i \cdot c^2 \cdot Vou_i \qquad Emou_i := \rho m_i \cdot c^2 \cdot Vou_i \qquad E\Lambda ou_i := \rho\Lambda_i \cdot c^2 \cdot Vou_i$$

$$M_{\rho v_i} := \rho m_i \cdot V_i \qquad Er_i := \rho r_i \cdot c^2 \cdot V_i \qquad Em_i := \rho m_i \cdot c^2 \cdot V_i \qquad E\Lambda_i := \rho\Lambda_i \cdot c^2 \cdot V_i$$

$$Eou := Erou + Emou + E\Lambda ou \qquad E := Er + Em + E\Lambda$$

Radiation - Matter Equality $\quad arm := \dfrac{\rho r_{700}}{\rho m_{700}}$ $\qquad$ **Matter - Lambda Equality** $\quad am\Lambda := \sqrt[3]{\dfrac{\rho m_{2600}}{\rho\Lambda_{2600}}} \qquad am\Lambda = 0.774$

$$ar_i := \sqrt[4]{4\cdot\Omega_{r0}}\cdot\left(H_0\cdot t_i\right)^{\frac{1}{2}} \qquad am_i := \sqrt[3]{2.25\cdot\Omega_{m0}}\cdot\left(H_0\cdot t_i\right)^{\frac{2}{3}} \qquad a\Lambda_i := am\Lambda\cdot e^{\sqrt{1-\Omega_{m0}}\cdot H_0\cdot t_i}$$

$$C_{inf} = 8\pi G\cdot\frac{f}{3} + \frac{\Lambda}{3} \qquad a_{inflation}(t) = e^{\sqrt{C_{inf}}\cdot t} \qquad One\_Year := 3600\cdot 24\cdot 365 \qquad Now := t_{100\cdot OM}\cdot s^{-1}$$

Plots of the Ratio of Lookback time to H0 and the Ratio of Time to H0

$$tL\_tH0\left(z,\Omega_{0m},\Omega_{0\Lambda},\Omega_{0r}\right) := \int_0^z \frac{1}{(1+z\xi)\cdot\sqrt{\Omega_{0m}\cdot(1+z\xi)^3 + \Omega_{0\Lambda} + \Omega_{0r}\cdot(1+z\xi)^4}}\,dz\xi$$

$$tL\_tH0(1000,0.1,0.7,0.2) = 0.767$$

$$t\_tH0\left(z,\Omega_{0m},\Omega_{0\Lambda},\Omega_{0r}\right) := \int_0^{(1+z)^{-1}} \frac{a}{\sqrt{\Omega_{0m}\cdot a + \Omega_{0\Lambda}\cdot a^4 + \Omega_{0r}}}\,da$$

$$t\_tH0(1000,0.1,0.7,0.2) = 1.116\times 10^{-6}$$

Comoving Distance

$$z = \frac{1}{a} - 1$$

$$D_{Cz}\left(z,\Omega_{0m},\Omega_{0\Lambda},\Omega_{0r}\right) := \int_{(1+z)^{-1}}^{1} \frac{1}{\sqrt{\Omega_{0m}\cdot a\xi + \Omega_{0\Lambda}\cdot a\xi^4 + \Omega_{0r}}}\,da\xi$$

Apparent Magnitude-Redshift Relation (Mukhanov) Eq 2.81 (See Section X of this Paper)

For Comoving Distance, χ_{em}

$$\chi = \int_{t_{em}}^{t_0}\frac{dt}{a(t)} \qquad \chi_{em}\left(z,\Omega_m\right) := \int_0^z \frac{1}{\sqrt{\Omega_m\cdot(1+z\xi)^3 + \left(1-\Omega_m\right)}}\,dz\xi$$

$$\Phi^2\left(\chi_{em}\right) = \begin{cases} \sinh^2\chi, & k=-1; \\ \chi^2, & k=0; \\ \sin^2\chi, & k=+1. \end{cases}$$

photon emitted at time t_{em}

Note: For k = 0

$$\Phi\left(\chi_{em}\right) = \chi_{em}$$

Bolometric Flux is the Flux Integrated over Entire Spectrum
Then the Bolometric Magnitude for k=0 is Given by:

$$m_{bol}\left(z,\Omega_m\right) := 5\,log(1+z) + 5\,log\left(\chi_{em}\left(z,\Omega_m\right)\right) + 25$$

VIII. Multiple-Component Universes: Parameter (t_0H_0) Contour Vs. Densitites

ASTROPHYSICS AND COSMOLOGY
Juan Garcıa-Bellido, Theoretical Physics Group

Define $y = \frac{a}{a_0}$ $\qquad \tau = H_0 \cdot (t - t_0)$

Then Friedmann's Equation can be written:

$$\frac{d}{d\tau} y = \sqrt{1 + \left(\frac{1}{y} - 1\right) \cdot \Omega_M + \left(y^2 - 1\right) \cdot \Omega_\Lambda} \qquad \text{Equation 56}$$

With Initial Conditions

$$y(0) = 1 \qquad \frac{d}{d\tau} y(0) = 1$$

Therefore, the present age t_0 is a function of the other parameters,
$t_0 = f(H_0, \Omega_M, \Omega_\Lambda)$, determined from

$$t0H0\left(\Omega_M, \Omega_\Lambda\right) := \int_0^1 \frac{1}{\sqrt{1 + \left(\frac{1}{y} - 1\right) \cdot \Omega_M + \left(y^2 - 1\right) \cdot \Omega_\Lambda}} \, dy \qquad t0H0(0.3, 0.7) = 0.964$$

Calculate a Matrix $Time_0H_0$ (t0H0) of Values:

of t0H0 for Ω_M and Ω_Λ Ranging from 0 to 1.5

```
Time0H0 := | TML ← (0  0  0)
           | m ← 0
           | l ← 0
           | for mm ∈ 0, 1 .. 170
           |   | m ← m + 0.01
           |   | l ← 0
           |   | for ll ∈ 0, 1 .. 100
           |   |   | th ← t0H0(m, l)
           |   |   | tml ← (m  l  th)
           |   |   | l ← l + 0.01
           |   |   | TML ← stack(TML, tml)
           | TML
```

$min\left(Time0H0^{\langle 0 \rangle}\right) = 0$

$min\left(Time0H0^{\langle 1 \rangle}\right) = 0$

$min\left(Time0H0^{\langle 2 \rangle}\right) = 0$

$max\left(Time0H0^{\langle 0 \rangle}\right) = 1.71$

$max\left(Time0H0^{\langle 1 \rangle}\right) = 1$

$max\left(Time0H0^{\langle 2 \rangle}\right) = 2.062$

$rows(Time0H0) = 1.727 \times 10^4$

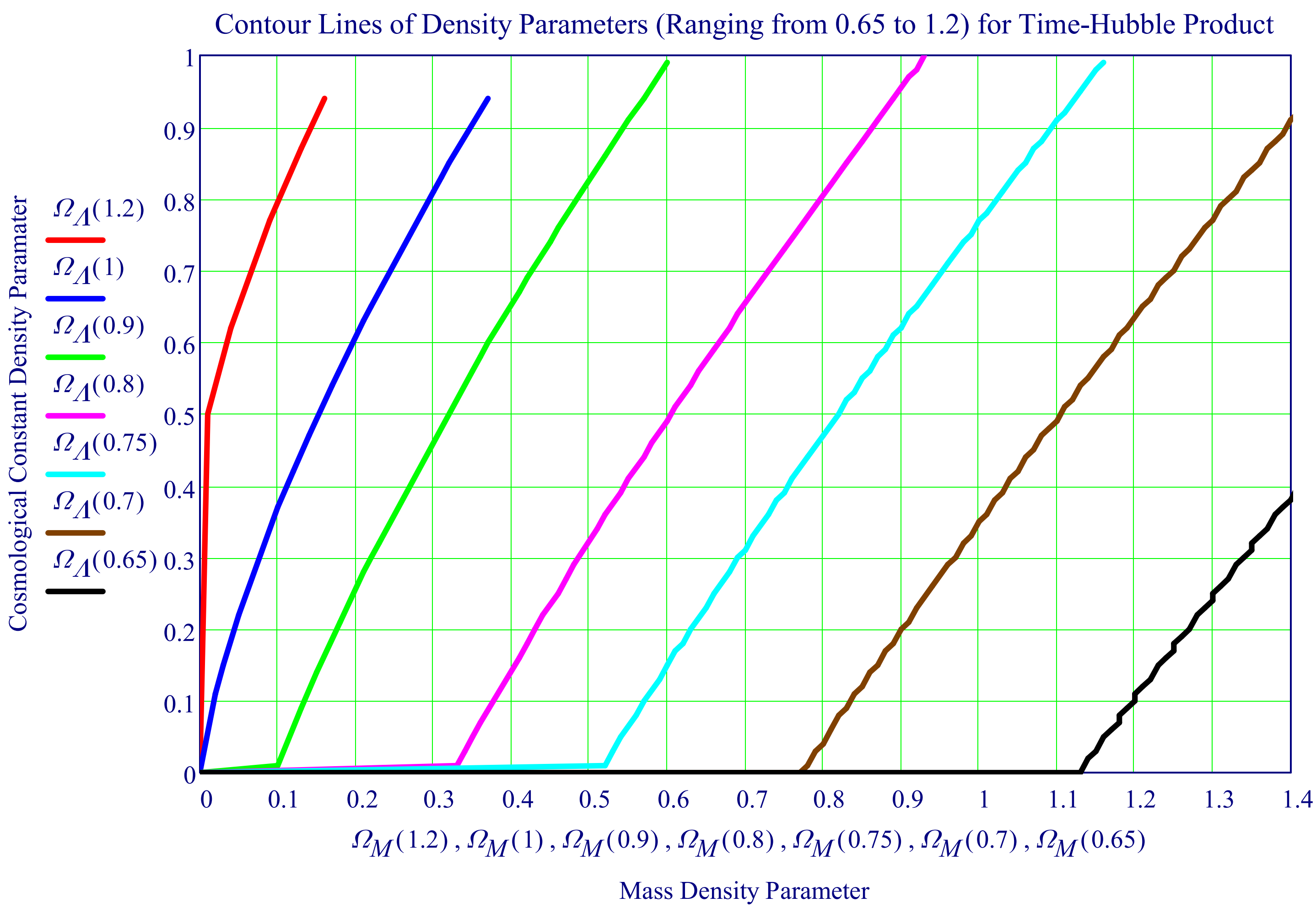
Assemble Contour Line Points of Curves with Given t_0H_0 Values

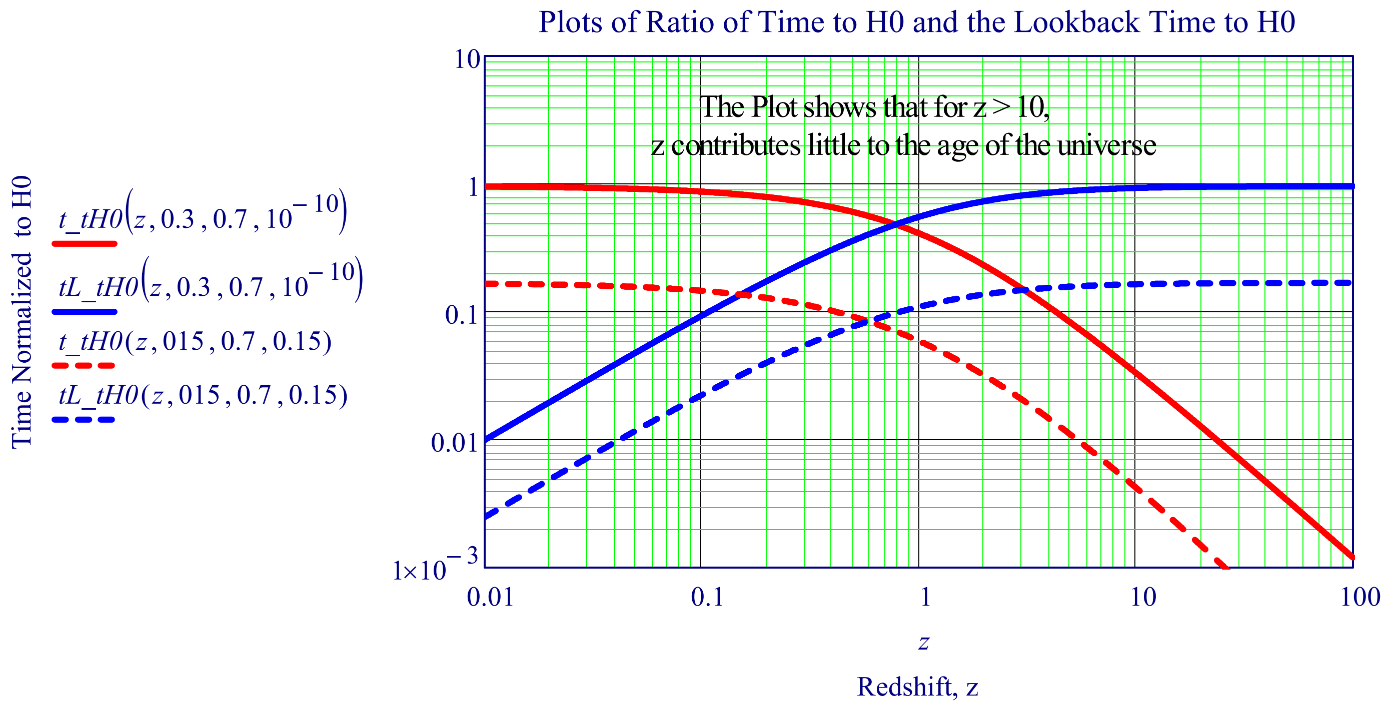
Find Those Contour Values of **Density Parameters**, Ω_M and Ω_Λ,
of Matrix $Time_0H0$ that Give a t_0H_0 values (T) ranging from 0.65, 0.7 ... up to 1.2

$$
TH(T) := \left|
\begin{array}{l}
R \leftarrow 0 \\
TH \leftarrow (0 \;\; 0 \;\; 0) \\
for \;\; r \in 0, 1 \,..\, 17000 \\
\quad if \;\left(Time0H0_{r,2} < T + 0.001\right) \wedge Time0H0_{r,2} > T - 0.00 \\
\quad\quad \left| \begin{array}{l} out \leftarrow \left(Time0H0_{r,0} \;\; Time0H0_{r,1} \;\; Time0H0_{r,2}\right) \\ TH \leftarrow stack(TH, out) \end{array} \right. \\
TH
\end{array}
\right.
$$

$$\Omega_\Lambda(T) := TH(T)^{\langle 1 \rangle} \qquad \Omega_M(T) := TH(T)^{\langle 0 \rangle} \qquad t0H0(1,0) = 0.667$$

$$t0H0(0.01,1) = 2.062$$





2023 Estimate z=10 is 13.30 Gyr

$t_{BB} := 13.8 Gyr$

$t_{BB} \cdot tL\_tH0\left(10, 0.3, 0.7, 10^{-10}\right) = 12.844 \cdot Gyr$ $\qquad z = \frac{1}{a} - 1$

Dynamics of the expansion

To the observer, the evolution of the scale factor is most directly characterized by the change with redshift of the Hubble parameter and the density parameter; the evolution of H (z) and Ω(z) is given immediately by the Friedmann equation in the form $H^2 = 8\pi G\rho/3 - kc^2/R^2$. Inserting the above dependence of **ρ** on ***a*** gives

$$\boxed{H^2(a) = H_0^2\left[\Omega_v + \Omega_m a^{-3} + \Omega_r a^{-4} - (\Omega - 1)a^{-2}\right].}$$

This is a crucial equation, which can be used to **obtain the relation between redshift and comoving distance.** The radial equation of motion for a photon is $R\,dr = c\,dt = c\,dR/R_{dot} = c\,dR/(RH)$.

With $R = R_0/(1+z)$, this gives

$$\begin{aligned} R_0 dr &= \frac{c}{H(z)}\,dz \\ &= \frac{c}{H_0}\left[(1-\Omega)(1+z)^2 + \Omega_v + \Omega_m(1+z)^3 + \Omega_r(1+z)^4\right]^{-1/2} dz. \end{aligned}$$

<u>This relation is arguably the single most important equation in cosmology,</u>
since it shows how to relate comoving distance to redshift, Hubble constant and density parameter.

The comoving distance determines the apparent brightness of distant objects, and the comoving volume element determines the numbers of objects that are observed. These aspects of observational cosmology are discussed in more detail below.

Lastly, using the expression for H(z) with $\Omega(a) - 1 = kc^2/(H^2 R^2)$ gives

the redshift dependence of the total density parameter:

$$\boxed{\Omega(a) - 1 = \frac{\Omega - 1}{1 - \Omega + \Omega_v a^2 + \Omega_m a^{-1} + \Omega_r a^{-2}}.}$$

This last equation is very important.

It tells us that, at high redshift, all model universes apart from those with only vacuum energy will tend to look like the $\Omega = 1$ model.

This is not surprising given the form of the Friedmann equation: provided $\rho R^2 \rightarrow \infty$ as $R \rightarrow 0$, the $-kc^2$ curvature term will become negligible at early times.
If $\Omega \neq 1$, then in the distant past $\Omega(z)$ must have differed from unity by a tiny amount: the density and rate of expansion needed to have been finely balanced for the universe to expand to the present.

This tuning of the initial conditions is called the flatness problem and is one of the motivations for the applications of quantum theory to the early universe.

Evolution of the Hubble Factor: Mass Conservation of non-relativistic matter implies $\rho_m \propto a^{-3} = (1+z)^3$.
In the ΛCDM model, dark energy is assumed to behave like a cosmological constant: $\rho_\Lambda \propto a^0 = (1+z)^0$.
The density of radiation (and massless neutrinos) scales as $\rho_r \propto a^{-4} = (1+z)^4$ because the number density of photons is $\propto a^{-3} = (1+z)^3$ and the mass $E/c^2 = h\nu/c^2$ of each photon scales as $E \propto \lambda^{-1} \propto (1+z)^1 \propto a^{-1}$.

$\Omega_{r0} = 8.7 \times 10^{-5}$

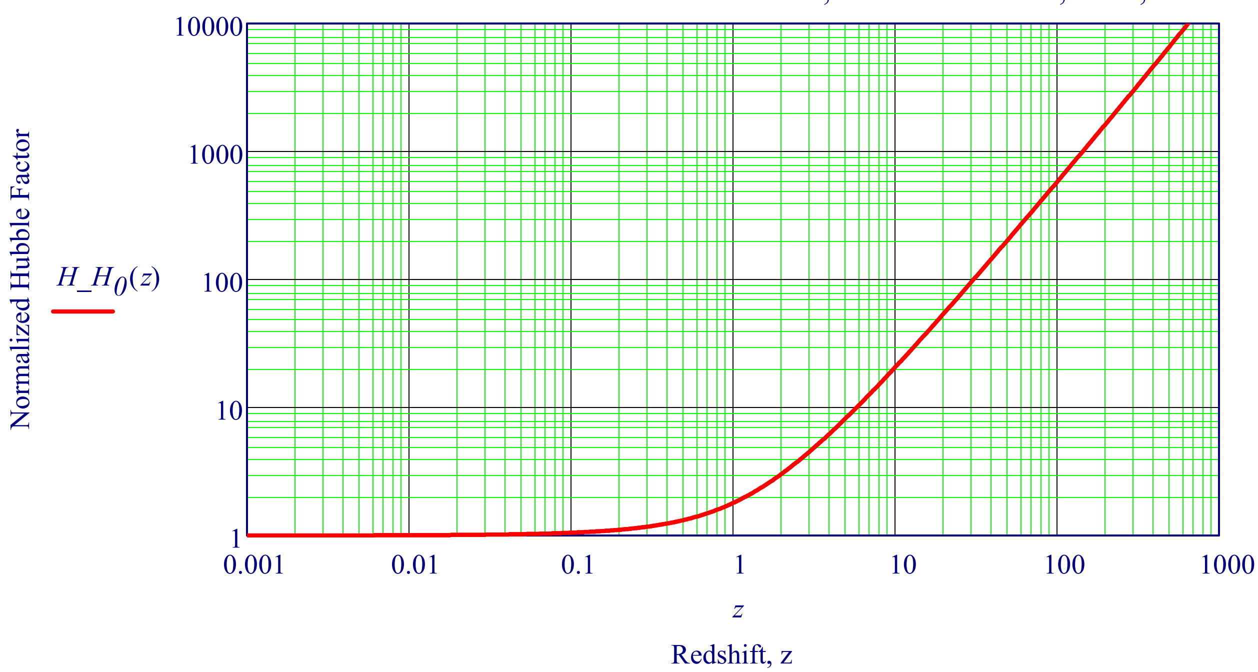
$\Omega_{m0} = 0.317$

$\Omega_{\Lambda0} = 0.683$

Dynamical Equation Specifying the Evolution of the Hubble Factor of Our Universe

$$\frac{H}{H_0} = H\_H_0(z) := \sqrt{\Omega_{m0}\cdot(1+z)^3 + \Omega_{\Lambda0} + \Omega_{r0}\cdot(1+z)^4}$$

Evolution of the Hubble Factor vs. Redshift, z for Given Ωm0, ΩΛ0, Ωr0

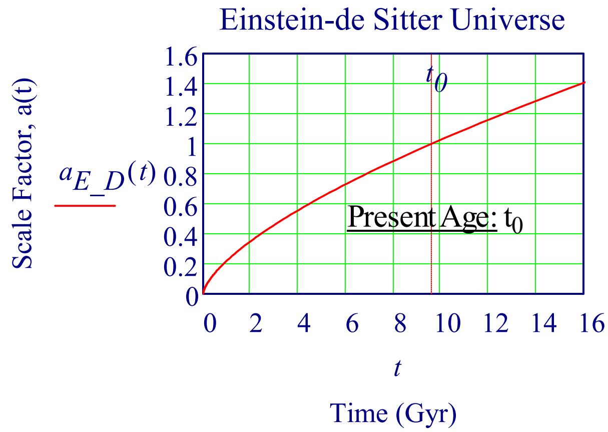


Einstein-de Sitter (EdS) Universe is a Flat and Matter-Only Universe

The Einstein–de Sitter universe is a model of the universe proposed by Albert Einstein and Willem de Sitter in 1932. On first learning of Edwin Hubble's discovery of a linear relation between the redshift of the galaxies and their distance, **Einstein set the cosmological constant to zero** in the Friedmann equations, resulting in a model of the expanding universe known as the Friedmann–Einstein universe. In 1932, Einstein and De Sitter proposed an even simpler cosmic model by assuming a **vanishing spatial curvature as well as a vanishing cosmological constant**. In modern parlance, the Einstein–de Sitter universe can be described as a **cosmological model for a flat matter-only** Friedmann–Lemaître–Robertson–Walker metric (FLRW) universe.

In the model, Einstein and de Sitter derived a simple relation between the average density of matter in the universe and its expansion according to $H_0^2 = \kappa\rho/3$, where H_0 is the Hubble constant, ρ is the average density of matter and κ is the Einstein gravitational constant. The cosmic time t as a function of scale factor, a, is giv

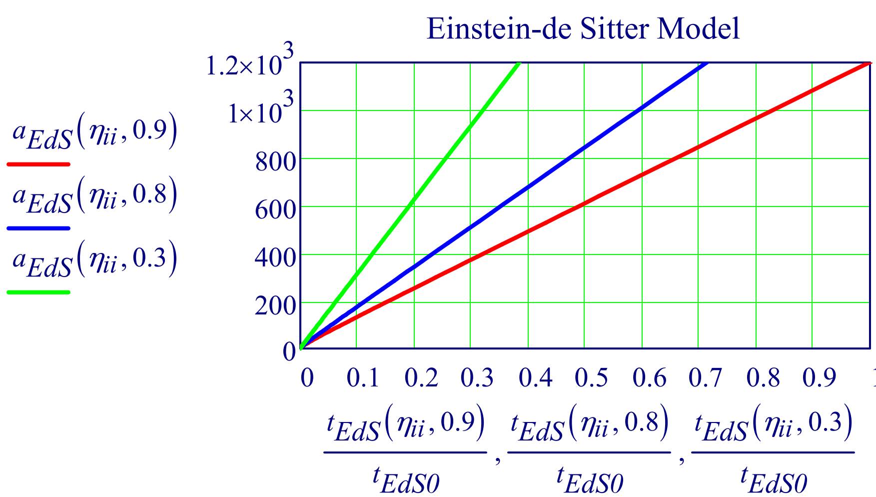
$$a_{eds}(t) := c\cdot e^{\sqrt{\frac{8\pi\cdot G\cdot \rho_0}{3}}\cdot t} \qquad t_0 := \frac{2}{3\cdot H_0} = 9.612\cdot Gyr \qquad t_0 := \frac{t_0}{Gyr} \qquad a_{E\_D}(t) := \left(\frac{t}{t_0}\right)^{\frac{2}{3}}$$



EdS: The cosmic time t as a function of the scale factor, a, is given by the Expression:

$$a_{EdS}(\eta, \Omega_0) := \frac{1}{2}\cdot\frac{\Omega_0}{1-\Omega_0}\cdot(\cosh(\eta) - 1) \qquad t_{EdS}(\eta, \Omega_0) := \frac{1}{2H_0\cdot\frac{Gyr\cdot km}{Mpc}}\cdot\frac{\Omega_0}{(1-\Omega_0)^{\frac{3}{2}}}\cdot(\sinh(\eta) - \eta)$$

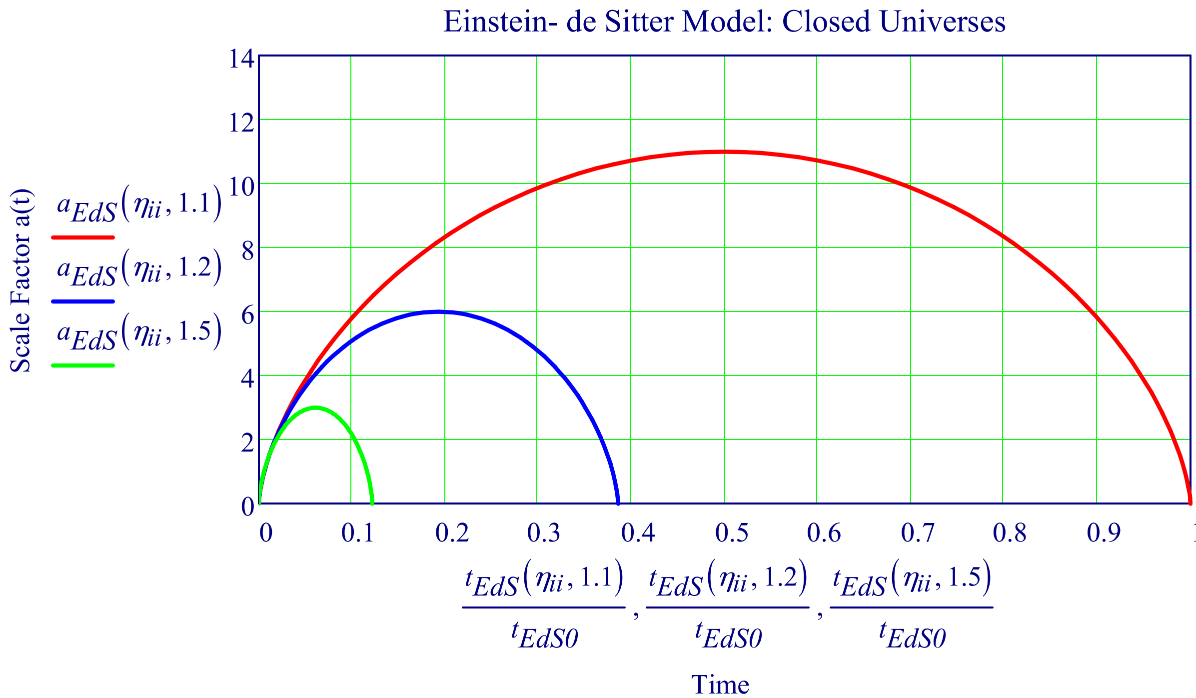
$ii := 0..200$ $\qquad \eta_{ii} := \frac{2\cdot\pi\cdot ii}{100}$ $\qquad$ Time Normalized to $\Omega_0 = 0.9$ $\qquad t_{EdS0} := t_{EdS}(2\pi, 0.9)$



Plots of a(t) versus t for the closed universes with Ω_0 = 1.1, 1.2, 1.5,

$$a_{EdS}(\eta,\Omega_0) := \frac{1}{2}\cdot\frac{\Omega_0}{\Omega_0-1}\cdot(1-\cos(\eta)) \qquad t_{EdS}(\eta,\Omega_0) := \frac{1}{2H_0\cdot\frac{Gyr\cdot km}{Mpc}}\cdot\frac{\Omega_0}{(\Omega_0-1)^{\frac{3}{2}}}\cdot(\eta-\sin(\eta))$$

$$ii := 0..100 \qquad \eta_{ii} := \frac{2\cdot\pi\cdot ii}{100} \qquad t_{EdS0} := t_{EdS}(2\pi, 1.1)$$

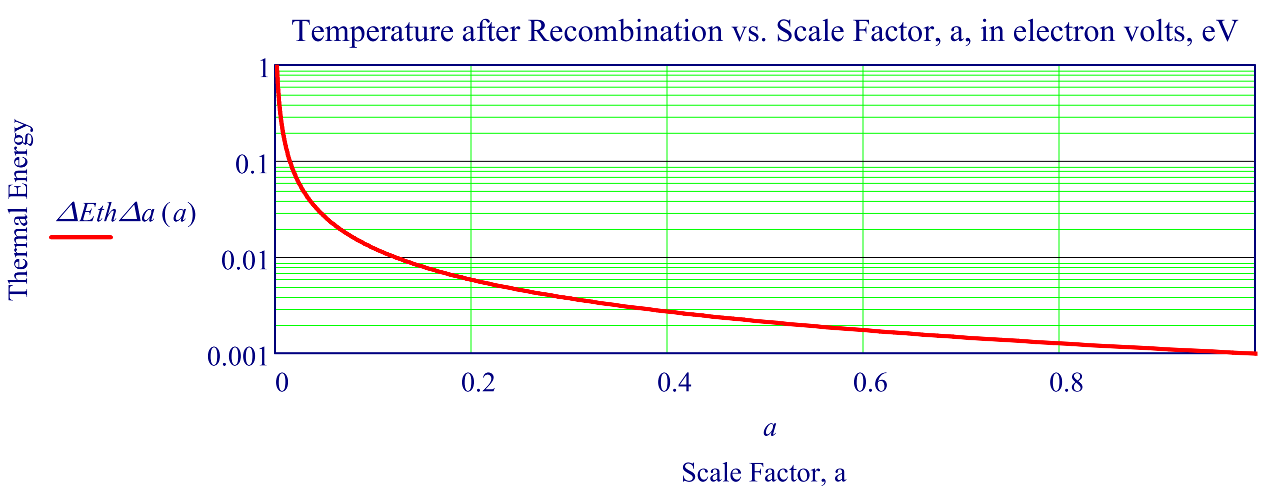


Temperature Jumps at Phase Transitions. Temperature at Recombination, E_{th} .

A New Version of the Lambda-CDM Cosmological Model, with Extensions and New Calculations, Journal of Modern Physics, 2024, 15, 193-238, Jan Helm

Rate of Change of Eth with scale factor a, ΔEthΔa

$$\Delta Eth\Delta a = \frac{d}{da}Eth \qquad \Delta Eth\Delta a(a) := \frac{-E_{th0}}{a^{\eta}}$$



IX. Stellar Classification Systems - MK, Harvard, Hertzsprung–Russell

Luminosity Defins. - Absolute & Apparent Magnitudes, Distance Modulus, Luninous Flux

Magnitude, in astronomy, is a measure of the **brightness** of a star or other celestial body.
The distance modulus, μ, is a **way of expressing distances** that is often used in astronomy. It describes distances on a **logarithmic scale** based on the astronomical magnitude system. The **apparent magnitude, m**, of a star is the magnitude it has **as seen by an observer on Earth.** The distance modulus, μ, is defined as $\mu = m - M$ (ideally, corrected from the effects of interstellar absorption) where **M**, is the a**bsolute magnitude,** of an astronomical object.

Luminous flux is a measure of the **power of visible light** produced by a light source or light fitting. It is measured in lumens (lm). **Luminosity**, in astronomy, the amount of light emitted by an object in a unit of time, or its power (W). The luminosity of the Sun is 3.846×10^{26} watts. Luminosity is an absolute measure of radiant power; that is, its value is independent of an observer's distance from an object

Irradiance (or flux density) is a term of radiometry and is defined as the radiant flux received by some surface per unit area. In the SI system, it is specified in units of W/m^2.

Absolute magnitude M is defined as the apparent magnitude of an object when seen at a **distance of 10 parsecs**. If a light source has luminosity L(d) when observed from a distance of d parsecs, and luminosity L(10) when observed from a distance of 10 parsecs, the inverse-square law is then written like:

$$L(d) = \frac{L(10)}{\left(\frac{d}{10}\right)^2}$$

The apparent m and absolute magnitude M and flux, F(d), are related by:

$$m = -2.5 \log_{10} F(d)$$

$$M = -2.5 \log_{10} F(d = 10)$$

Estimating Distance to Star from Apparent Brightness and Hertzprung-Russell Diagram

One can use detailed observations of nearby stars to provide a means to measure distances to more distant stars. Using spectroscopy, one can measure precisely the colour of a nearby star; using photography, one can also measure its apparent brightness.

Using the apparent brightness, m, the distance, and inverse square law, one can **compute the absolute brightness** of these stars. Ejnar Hertzsprung (1873-1967) and Henry Russell (1877-1957) plotted this **absolute brightness against colo**r for thousands of nearby stars in 1905-1915. This yields the famous Hertzprung-Russell diagram. See Section IX. Once one has this diagram, one can use it in reverse to measure distances to more stars than parallax methods can reach. For any star, one can measure its colour and its apparent brightness and from the Hertzprung-Russell diagram, one can then infer the absolute brightness. From the apparent brightness and absolute brightness, one can solve for distance.

The distance modulus can be used to determine the distance to a star using the equation:
$M = m - 5 \log(d/10)$

Magnitudes of Some Cosmological Light Sources

Sun	-26.5
Full Moon	-12.5
Venus	-4.3
Mars or Jupiter	-2
Sirius (α CMa)	-1.44
Vega (α Lyr)	0.0
Alnair (α Gru)	1.73
Naked-eye limit	6.5
Binocular limit	10
Proxima Cen	11.09
Visual limit through 20 cm telescope	14
QSO at redshift $z = 2$	≈ 20
Cepheid in galaxy M100 observed with HST	26
Galaxy at $z = 6$ observed with Gemini 8.1 m telescope	28
Limit for James Webb Space Telescope	≥ 30

Luminosity Distance

In relativistic cosmologies, observed flux (bolometric, or in a finite bandpass) is:

$$f = L \,/\, [\, (4\pi\, D^2)\, (1+z)^2\,]$$

One factor of $(1+z)$ is due to the energy loss of photons, and one is due to the time dialation of the photon rate.

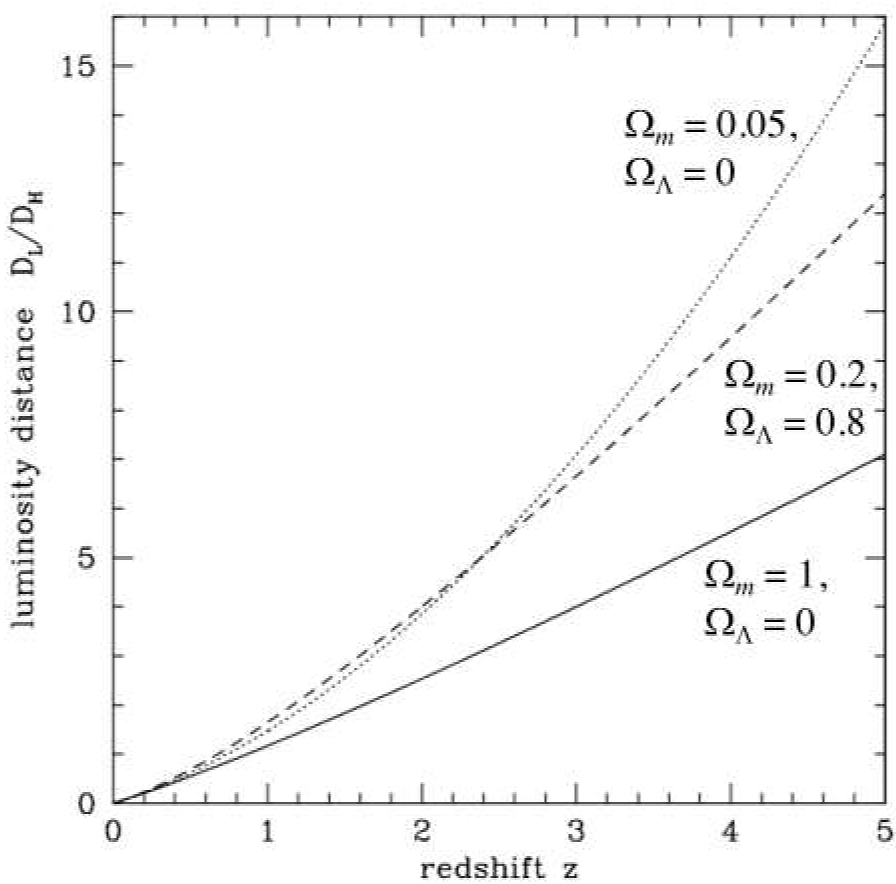
A **luminosity distance** is defined as $D_L = D\,(1+z)$, so that $f = L \,/\, (4\pi\, D_L^2)$.

For a specific flux, however,

$$S_\lambda = \frac{1}{(1+z)} \frac{L_{\lambda/(1+z)}}{L_\lambda} \frac{L_\lambda}{4\pi\, D_{\mathrm{L}}^2}$$

(since Angstroms are also stretched by 1+z)

Luminosity Distance

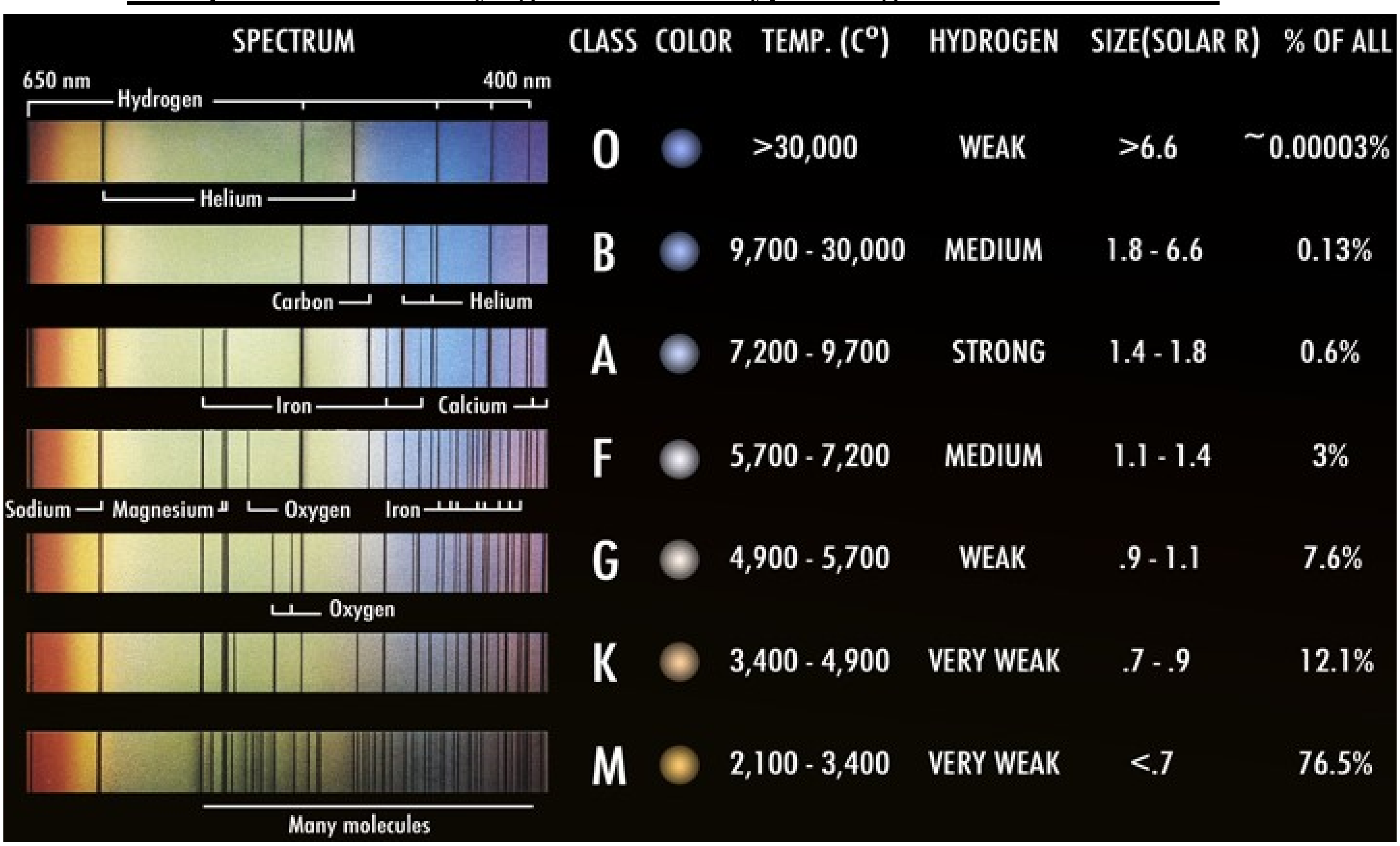


Stellar Classification Systems - MK, Harvard, Hertzsprung–Russell

Wikipedia - "In astronomy, stellar classification is the classification of stars based on their **spectral characteristics**. Electromagnetic radiation from the star is analyzed by splitting it with a prism or diffraction grating into a spectrum exhibiting the rainbow of colors interspersed with spectral lines. Each line indicates a particular chemical element or molecule, with the line strength indicating the abundance of that element. The strengths of the different spectral lines vary mainly due to the temperature of the photosphere, although in some cases there are true abundance differences. The spectral class of a star is a short code primarily summarizing the ionization state, giving an objective measure of the photosphere's temperature.
Most stars are currently classified under the Morgan–Keenan (MK) system using the letters O, B, A, F, G, K, and M, a sequence from the hottest (O type) to the coolest (M type). Each letter class is then subdivided using a numeric digit with 0 being hottest and 9 being coolest (e.g., A8, A9, F0, and F1 form a sequence from hotter to cooler). The sequence has been expanded with classes for other stars and star-like objects that do not fit in the classical system, such as class D for white dwarfs and classes S and C for carbon stars.
In the MK system, a luminosity class is added to the spectral class using Roman numerals. This is based on the width of certain absorption lines in the star's spectrum, which vary with the density of the atmosphere and so distinguish giant stars from dwarfs. Luminosity class 0 or Ia+ is used for hypergiants, class I for supergiants, class II for bright giants, class III for regular giants, class IV for subgiants, class V for main-sequence stars, class sd (or VI) for subdwarfs, and class D (or VII) for white dwarfs. The full spectral class for the Sun is then G2V, indicating a main-sequence star with a surface temperature around 5,800 K.

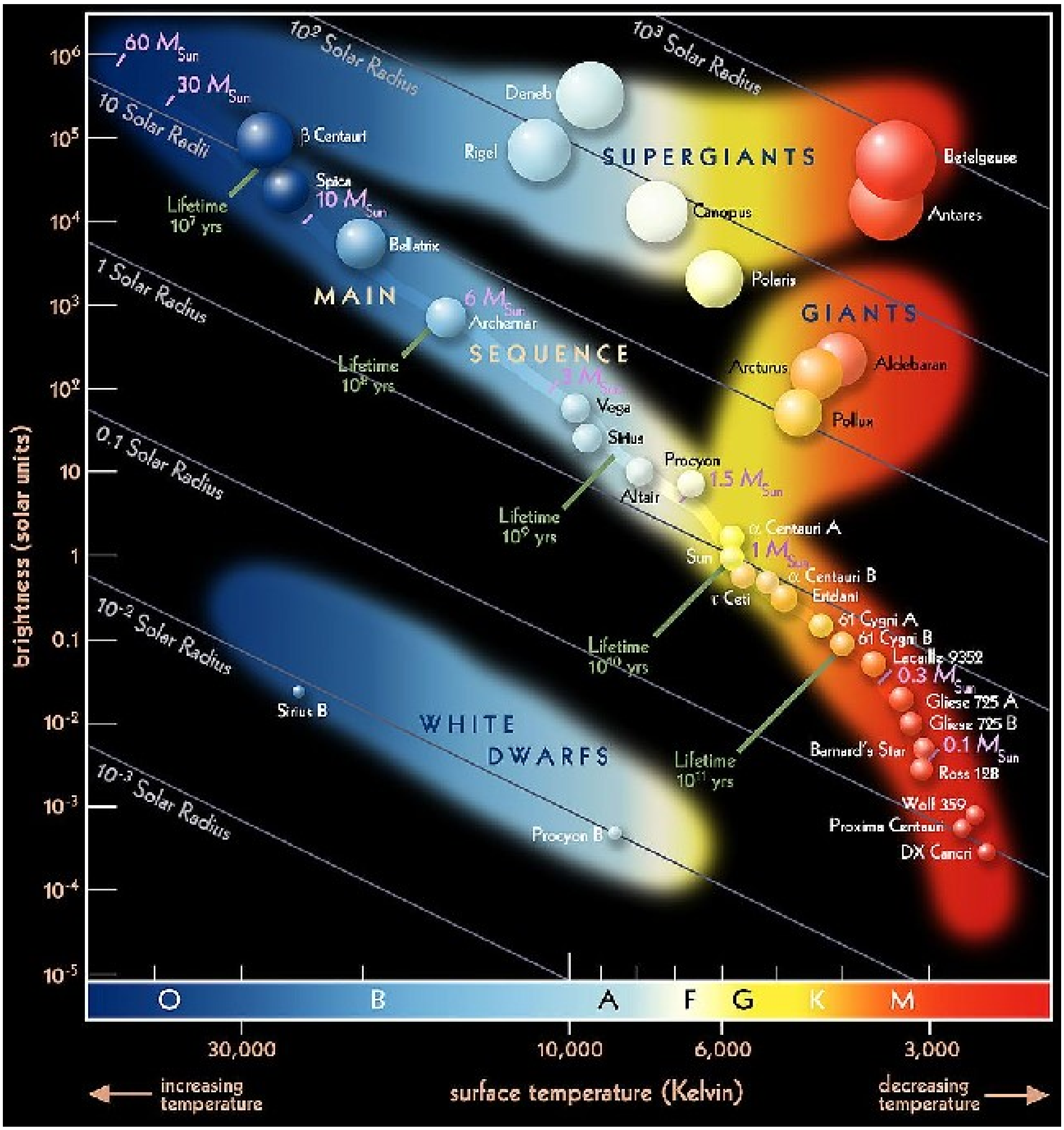
Harvard spectral classification
The Harvard system is a one-dimensional classification scheme by astronomer Annie Jump Cannon, who re-ordered and simplified the prior alphabetical system by Draper (see History). Stars are grouped according to their spectral characteristics by single letters of the alphabet, optionally with numeric subdivisions. Main-sequence stars vary in surface temperature from approximately 2,000 to 50,000 K, whereas more-evolved stars can have temperatures above 100,000 K[citation needed]. Physically, the classes indicate the temperature of the star's atmosphere and are normally listed from hottest to coldest."

A simple chart for classifying the main star types using Harvard classification



The Hertzsprung–Russell (H-R) diagram

Is a scatter plot of stars showing the relationship between the stars' absolute magnitudes or luminosities versus their stellar classifications or effective temperatures. The diagram was created in 1911 and represented a major step towards an understanding of stellar evolution. The H-R diagram is quite easy to understand if you can interpret what each axis means. The horizontal axis measures the surface temperature of the star in Kelvin. Stars on the right of the horizontal axis are cooler and redder in colour than the stars on the left, with temperatures of around 3000 Kelvin as opposed to 25,000 Kelvin upwards. The vertical axis on the left measures luminosity using the Sun as our comparison. So, a luminosity of one is equal to one Sun. The vertical axis on the right measure's absolute magnitude, or brightness, crucially considering a star's distance. The bottom axis identifies spectral type, or, spectral class of a star, which is another way to describe the colour and temperature. **Plotting Cepheids, RR Lyrae, Mira and Semiregular pulsating variable stars** on the H-R diagram is not a single plot like non-pulsating stars. During their evolution through the instability strips they are pulsationally unstable – expanding and brightening, then contracting and become dimmer. The instability strips for Miras and Cepheids are especially elongated because of these expansions and contractions. Some pulsating variable stars change in temperature by two spectral classes during one cycle from maximum to minimum. To show the entire cycle of change for individual variable stars, it is necessary to plot them twice on the H-R diagram – both at max and min absolute magnitude.



Spectral Analysis of Different Types of Stars

This Data Can be Obtained from Table Top Equipment to Determine the Type of Star and Redshift, z:
Celestron NEXSTAR 8SE COMPUTERIZED TELESCOPE. A diffraction grating with grooves that are spaced at 100 lines/mm, mounted in a standard 1.25″ filter cell and attached to a DSLR camera.

Main Sequence Star Types by Temperature

Type	Absorption lines	Temperature	Example
O	(H I, He I,) He II, N III , O III, Si IV	> 30000	
B	H I, He I, O II,Si III	> 10000	Orion's Belt
A	**H I**, Mg II,Si II, (Fe II, Ti II, Ca II)	> 7000	Sirius
F	H I, Ca II, Fe I, Ti I, Fe II, Ti II	> 6000	Procyon
G	(H I,) **Ca II**, Fe I, Ti I, etc., CH	> 5300	Sun
K	Ca II, **Ca I**, etc., TiO	> 4000	Arcturus
M	Ca I, TiO, etc.	> 2000	Betelgeuse

Get Star Data From PV Light House Spectral Irradiance Measurement Library

https://www2.pvlighthouse.com.au/resources/optics/spectrum%20library/spectrum%20library.aspx

B Type Star Spectral Irradiance Measurements

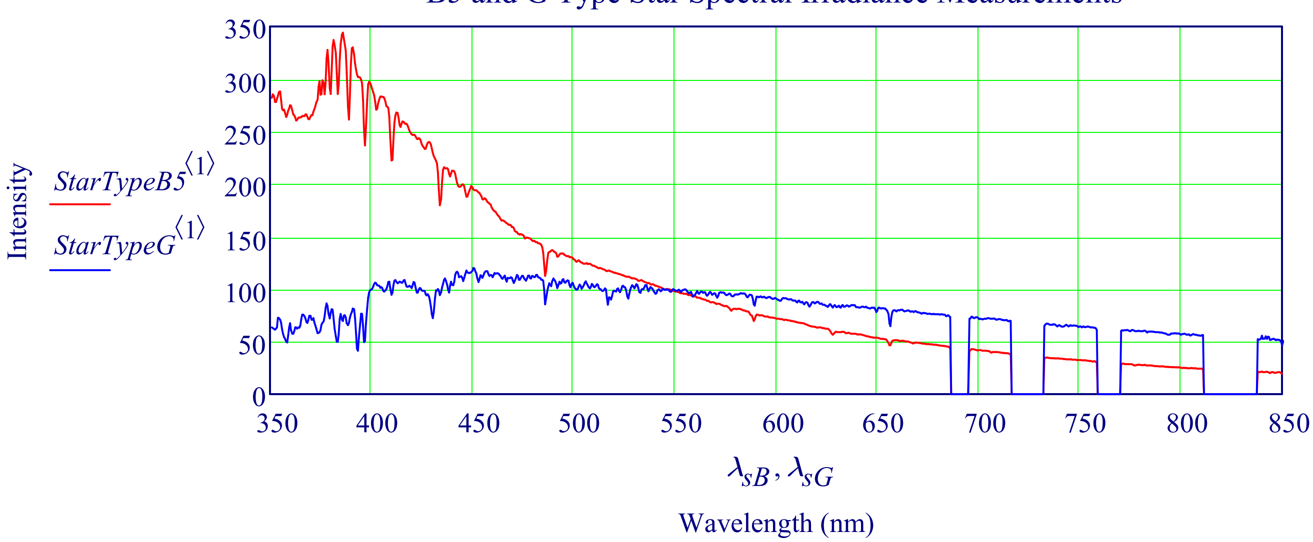
$StarTypeB5 := READPRN(\text{"B5 Star Spectrum.txt"})$ $\lambda_{sB} := StarTypeB5^{\langle 0\rangle}$

G Type Star Spectral Irradiance Measurement

$StarTypeG := READPRN(\text{"G Star Spectrum3.txt"})$ $\lambda_{sG} := StarTypeG^{\langle 0\rangle}$

Note: For these particular Type B and G stars, the peaks are consistent with Type, but shapes are different.

B5 and G Type Star Spectral Irradiance Measurements



G Type Star (Sun) Spectral Irradiance Data - Sun AM0 & AM1.5

Get Star Data From Spectrum Library

https://www2.pvlighthouse.com.au/resources/optics/spectrum%20library/spectrum%20library.aspx

AM0 and AM1.5 Correspond to the Sunlight at the Top of Atmosphere and at Sea Level, Respectively.

$SolarSpec_0 := READPRN(\text{"Solar AM0 Spectrum 280 -2500 2nm.txt"})$ $\quad SS_0 := SolarSpec_0$

$SolarSpec_{1.5} := READPRN(\text{"Solar AM1-5g Spectrum 280 -2500 2nm.txt"})$ $\quad SS_{1.5} := SolarSpec_{1.5}$

Planck's Spectral Radiation Law, B(λ,T)

$h := 6.6260693\cdot 10^{-34}\cdot joule\cdot sec \quad k_b := 1.3806505\cdot 10^{-23}\cdot\frac{joule}{K} \quad \lambda_s := SolarSpec_0^{\langle 0\rangle}$

$$B(\lambda,T) := \frac{2h\cdot c^2}{(nm\cdot\lambda)^5}\cdot\frac{1}{e^{\frac{h\cdot c}{nm\cdot\lambda\cdot k_b\cdot T}}-1}$$

$T_{sun} := 5777K$

Normalize Units B(λ,T): $Units := 2\cdot B(500, T_{sun})^{-1} \quad B_N(\lambda) := B(\lambda, T_{sun})\cdot Units$

Find Peak Wavelength for the AM0 Sun from its Blackbody Spectrum

$max\left(SolarSpec_0^{\langle 1\rangle}\right) = 2.075 \quad match\left(max\left(SolarSpec_0^{\langle 1\rangle}\right), SolarSpec_0^{\langle 1\rangle}\right) = (91)$

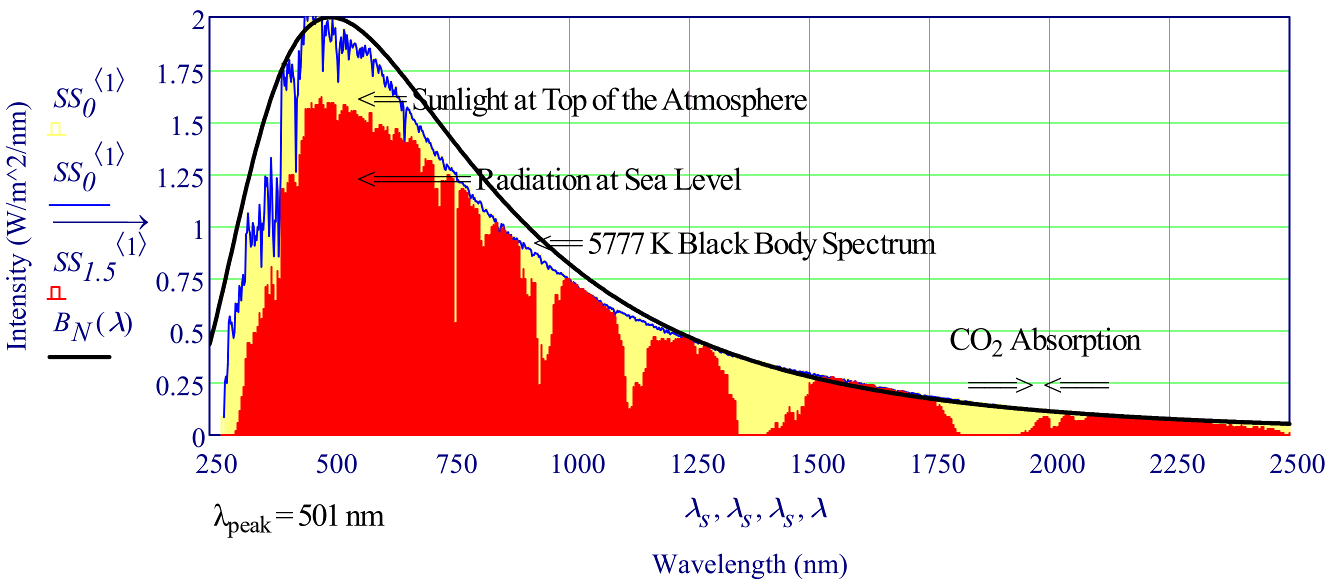
$\left(SolarSpec_0^{\langle 0\rangle}\right)_{91} = 462 \quad \lambda_{peak} := 462 \quad B_N(462) = 1.966$

The Sun's peak wavelength is between 483-504 nm (Green)

Wien's Displacement Law: Peak Wavelength Law

$\lambda_{max}(T) := \frac{0.2898cm\cdot K}{T} \quad \lambda_{max}(T_{sun}) = 501.644\cdot nm$

G Type Star (Sun) Spectrum - Top of Atmosphere & Sea Level - H2O Absorption



X. Measurement of Cosmic Distances - The Standard Candle

MEASURING COSMOLOGICAL PARAMETERS

The current proper distance to a galaxy, $d_p(t_0)$, is not a measurable property

Since cosmology is ultimately based on observations, if we want to find the distance to a galaxy, we need some way of computing a distance from that galaxy's observed properties. Let's focus on the properties that we can measure for objects at cosmological distances. We can measure the flux of light, f, from the object, in units of watts per square meter. The complete flux, integrated over all wavelengths of light, is called the bolometric flux. (A bolometer is an extremely sensitive thermometer capable of detecting electromagnetic radiation over a wide range of wavelengths.)

Cosmologists would like to know the scale factor a(t) for the universe. For a model universe whose contents are known with precision, the scale factor can be computed from the Friedmann equation. Finding a(t) for the real universe, however, is much more difficult. **The scale factor is not directly observable**; it can **only be deduced** indirectly from the **imperfect and incomplete observations** that we make of the universe around us.

The Standard Candle

One way of using measured properties to assign a distance is the standard candle method. A standard candle is an object whose luminosity L is known. For instance, if some class of astronomical object had luminosities which were the same throughout all of space-time, they would act as excellent standard candles – if their unique luminosity L were known. For instance, if some class of astronomical object had luminosities which were the same throughout all of space-time, they would act as excellent standard candles – if their unique luminosity L were known. Nowadays, the bolometric apparent magnitude of a light source is defined in terms of the source's bolometric flux, m,

$$m \equiv -2.5\log_{10}(f/f_x)$$

Reference Flux: $f_x := 2.53\cdot 10^{-8}\cdot\frac{W}{m^2}$

where the reference flux f_x is set at the value $f_x = 2.53\times 10^{-8}$ watt m^{-2}. Thanks to the **negative sign** in the definition, a small value of m corresponds to a large flux f. For instance, the flux of sunlight at the Earth's location is f = 1367 watts m^{-2}; the Sun thus has a bolometric apparent magnitude of m = −26.8.

The bolometric absolute magnitude of a light source is defined as the apparent magnitude that it would have if it were at a luminosity distance of dL = 10 pc. Thus, a light source with luminosity L has a bolometric absolute magnitude, M. Luminosity of the sun: $L_\odot$

Reference Luminosity: $L_\odot := 3.846\cdot 10^{26} W$ $\quad L_x := 78.7\cdot L_\odot$ $\quad M \equiv -2.5\log_{10}(L/L_x)$

Since that is the luminosity of an object which produces a flux fx = 2.53×10^{-8} watt m^{-2} when viewed from a distance of 10 parsecs. The bolometric absolute magnitude of the Sun is thus M = 4.74.

Given the definitions of apparent and absolute magnitude, the relation between an object's **apparent magnitude, m,** and its absolute magnitude, M, can be written in the form

$$M = m - 5\log\left(\frac{d_l}{10pc}\right)$$

The **distance modulus** is defined as **m − M**, and is related to the luminosity distance by the relation

$$m - M = 5\log\left(\frac{d_l}{10pc}\right) + 25$$

Using standard candles to determine the Hubble constant is the method used by Hubble himself. The recipe for finding the Hubble constant is a simple one:

- Identify a population of standard candles with luminosity L.
- Measure the redshift z and flux f for each standard candle.
- Compute $dL = (L/4\pi f)^{1/2}$ for each standard candle.
- Plot cz versus dL.
- Measure the slope of the cz versus dL relation when z << 1; the slope gives H_0.

Initial Mass Function, IMF

The properties and evolution of a star are closely related to its mass. In astronomy, the initial mass function (IMF) is an empirical function that describes the initial distribution of masses for a population of stars during star formation. IMF not only describes the formation and evolution of individual stars, it also serves as an important link that describes the formation and evolution of galaxies. The mass of a star can only be directly determined by applying Kepler's third law into binary stars system. However, the number of binary systems that can be observed is low, thus not enough samples to estimate the initial mass function. Therefore, stellar luminosity function is used to derive a mass function (present-day mass function, PDMF) by applying mass–luminosity relation. the luminosity function requires accurate determination of distances, and the most straightforward way is by measuring stellar parallax within 20 parsecs from the earth. The IMF is often stated in terms of a series of power laws, where $\xi(m)\Delta m$, the number of stars with masses in the range m to m + dm within a specified volume of space, is proportional to $m^{-\alpha}$, where α is a dimensionless exponent.

Note: The vertical axis for the Initial Mass Function $\xi(m)$ is **SCALED** so that for m greater than $M_\odot$, it is $(m/M_\odot)^{-2.35}$

Edwin E. Salpeter (1955) was the first astrophysicist who attempted to quantify IMF by applying power law into his equations. ξ_0 is a constant relating to the local stellar density

$$\xi_0 := 1 \qquad \xi(m,\Delta m) := \xi_0 \cdot \left(\frac{m}{M_\odot}\right)^{-2.35} \cdot \left(\frac{\Delta m}{M_\odot}\right) \qquad \xi_S(m) := \xi_0 \cdot \left(\frac{m}{1}\right)^{-2.35}$$

Kroupa (2001)

$$\xi_K(m) := if\left[m < 0.08, m^{-0.3} \cdot 15, if\left[(m \ge 0.08) \wedge (m \le 0.5), 1.3 \cdot m^{-1.3}, m^{-2.35}\right]\right]$$

***Intro to Cosmology,* 2nd. Ed., Ryden 2016 Equation 7.3**

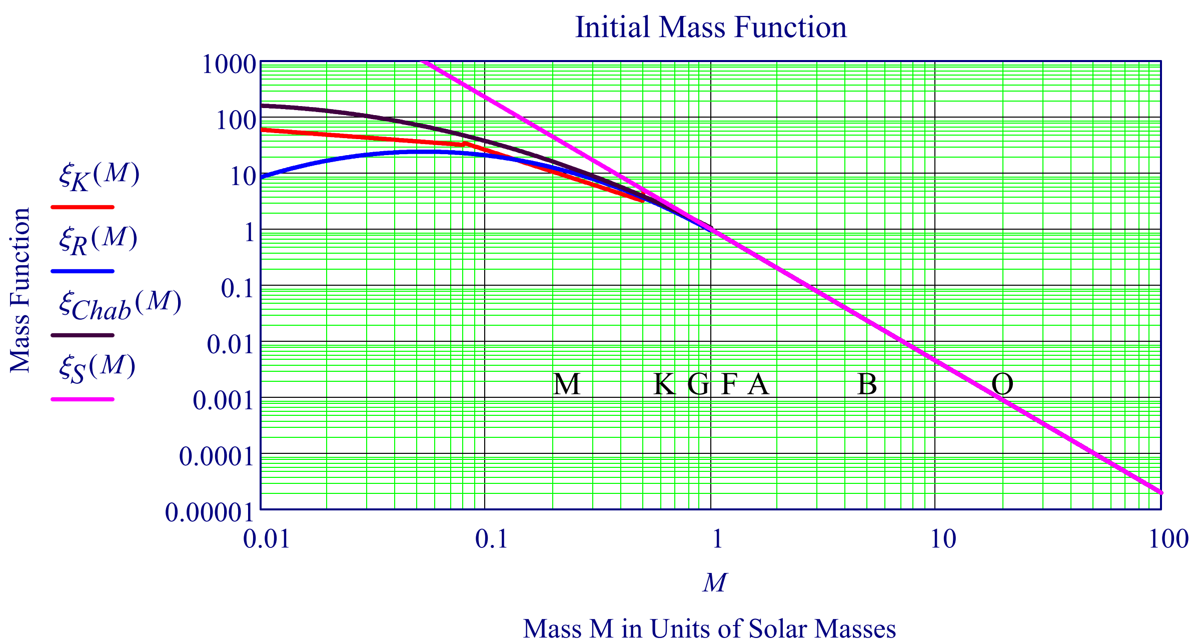
$$\xi_r(M) := 2.5 \cdot \frac{1}{M} \cdot exp\left[\frac{-(log(M) - log(0.2))^2}{2 \times 0.5^2}\right] \qquad \xi_R(M) := if\left(M \ge 1, M^{-2.35}, \xi_r(M)\right)$$

Chabrier (2003)

Chabrier gave the following expression for the density of individual stars in the Galactic disk, in units of parsec^{-3}

$$\xi_{chab}(M) := 55 \cdot \frac{0.158}{M \cdot ln0(10)} \cdot exp\left[\frac{-(log(M) - log(0.08))^2}{2 \times 0.69^2}\right] \qquad \xi_{Chab}(M) := if\left(M \le 1, \xi_{chab}(M), M^{-2.35}\right)$$

Mass ranges corresponding to the standard stellar spectral types O through M are indicated.



Initial Mass Function

Although O stars are extremely luminous, they are also short-lived. An O star with a mass $M = 60\ M_\odot$ will run out of fuel for fusion in a time $t \approx 3\,\mathrm{Myr}$; it will then explode as a type II supernova.

Note: $\Omega_{stars} = 0.3\%$

XI. Cosmic Distance Scale - Standard Candle 1: Cepheid Variables

The Standard Candle

To move outward in distance one starts One, with trigonometric parallaxes, then observes the same object with the other types of less precise parallaxes to calibrate and scale them. Once this is done one has the distance ladder reaching about 10,000 pc – halfway across the Milky Way. At this point one must put aside the parallax method and use other methods. With few exceptions, distances based on direct measurements are available only out to about a thousand pc, which is a modest portion of our own Galaxy. For distances beyond that, measurements are going to depend upon physical assumptions, that is, knowledge of the object in question. One must recognize the object and assume the class of objects is homogeneous enough that its members can be used for a meaningful estimation of distance – a standard candle as it were.

Almost all of the remaining rungs on the ladder are standard candles of one kind or another. A standard candle is an object that belongs to some class that has a known brightness (i.e., all members of the class have the same brightness). By comparing the known luminosity of the latter to its observed brightness, the distance to the object can be computed using the inverse square law.
Two problems exist for any class of standard candle. **The principal one is calibration**, determining exactly what the absolute magnitude of the candle is. This includes defining the class well enough that members can be recognized, and finding enough members with well-known distances that their true absolute magnitude can be determined with enough accuracy. The **second lies in recognizing members of the class**, and not mistakenly using the standard candle calibration upon an object which does not belong to the class. At extreme distances, which are where one most wishes to use a distance indicator, this recognition problem can be quite serious.

Standard Candle #1: Cepheid Variables

Cepheids were first noticed in 1784 in the constellation Cepheus in the northern sky, so these stars became known as "Cepheid variables." Cepheids are stars that periodically dim and brighten. In 1908 Henrietta Leavitt noticed a relationship between the brightness (or "luminosity") of a Cepheid variable star and its period for its pulsations in luminosity. They have a uniquie waveform and we can measure their period independent of how far away they are.
In the 1950s, astronomer Walter Baade discovered that the **nearby Cepheid variables used to calibrate the standard candle were of a different type than the more distant ones used to measure distances to nearby galaxies.** The nearby Cepheid variables were young, massive stars with much higher metal content than the distant old, faint ones. As a result, **the old stars were actually much brighter than believed,** and this had the ultimate effect of **doubling the distances** to the globular clusters, the nearby galaxies, and the diameter of the Milky Way.
Cepheids are luminous variable stars that **radially pulsate**. The strong direct relationship between a Cepheid's luminosity and its pulsation period makes them an important standard candle for Galactic and extragalactic sources. Type I Cepheids undergo pulsations with very regular periods on the order of days to months.
A relationship between the **period and luminosity for Type I Cepheids** was discovered in 1908 by **Henrietta Swan Leavitt** in her investigation of thousands of variable stars in the Magellanic Clouds. To use them as standard candles, one observes the pulsation period to get the luminosity (absolute magnitude). By then measuring the apparent brightness (value observed at Earth) one has everything needed to use the distance modulus m – M. The work was so important that Leavitt was considered for the Nobel Prize, but she died before her name could be submitted.

In addition, using data from the **HIPPARCOS astrometry satellite**, astronomers calculated the distances to many Galactic Cepheids using the trigonometric parallax technique. The resultant period-luminosity relationship for Type 1 Cepheids was:

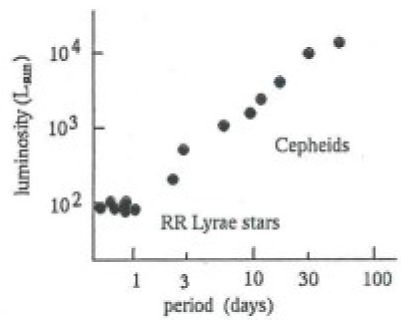
$$\mathbf{M_V = 2.81 \log(P) - (1.43 \pm 0.1)}$$

where $\mathbf{M_V}$ is the absolute magnitude and P is the period in days.

XII. Modeling the Dynamics of a Cepheid Variable

There are two classifications of variable stars, RRLyrae and Oepheid Variables. BB Lyrae have approximately a Solar mass and are yellow-white giants with luminosities on the order of 100 times that of the Sun. Cepheid Variables are yellow supergiants with several Solar masses and luminosities on the order of 20,000 times that of the Sun. These stars pulsate as the result of a special relationship between pressure and gravity. One idea is that as radiation emanates from the star, some of the He^{+} ionized into He^{2+} leading the surface of the star become more opaque.

As the surface darkens, less energy is able to escape therefore heating the gas within the star. As the gas heats it pushes outward expanding the staris radius. As the star grows in volume, the gas cools allowing the pressure inside to drop (He^{+2} converts back to He^{+}) and gravity to once again dominate by pulling everything inward. The cycle then is able to begin again.



Find The Period of a Cepheid Variable Star

From Newton's Second Law:

$$m \cdot \frac{d^2}{d\tau^2} R = \frac{-G \cdot M \cdot m}{R^2} + 4\pi R^2 \cdot P$$

In Equilibrium R is constant

$$\frac{G \cdot M \cdot m}{R^2} = 4\pi R^2 \cdot P$$

Let $R = R_0 + \delta R$ $\quad P = P_0 + \delta P$

$$4\pi R_0 \cdot P = \frac{G \cdot M \cdot m}{R_0^{\;3}}$$

$$m \cdot \frac{d^2}{d\tau^2}\left(R_0 + \delta R\right) = \frac{-G \cdot M \cdot m}{\left(R_0 + \delta R\right)^2} + 4\pi\left(R_0 + \delta R\right)^2 \cdot \left(P_0 + \delta P\right)$$

First Order Approximation:
(Taylor Series Expansion)

$$\frac{1}{\left(R_0 + \delta R\right)^2} = \frac{1}{R_0^{\;2}} \cdot \left(1 - 2 \cdot \frac{\delta R}{R_0}\right)$$

$$m \cdot \frac{d^2}{d\tau^2}(\delta R) = \frac{-G \cdot M \cdot m}{R_0^{\;2}} + \frac{2G \cdot M \cdot m}{R_0^{\;3}} + 4\pi R_0^{\;2} \cdot P_0 + 8\pi R_0 \cdot P_0 \cdot \delta R + 4\pi R_0^{\;2} \cdot \delta P$$

Substitute $\frac{G \cdot M \cdot m}{R^2} = 4\pi R^2 \cdot P$

$$m \cdot \frac{d^2}{d\tau^2}(\delta R) = \frac{2G \cdot M \cdot m}{R_0^{\;3}} + 8\pi R_0 \cdot P_0 \cdot \delta R + 4\pi R_0^{\;2} \cdot \delta P$$

For the adiabatic expansion of a gas: $\quad P_0 \cdot V_0^{\;\gamma} = P \cdot V^{\gamma} \quad P \cdot V^{\gamma} = Constant \quad V = \frac{4}{3}\pi \cdot R^3$

$$P \cdot R^{3\gamma} = Constant \qquad \frac{\delta P}{P_0} = -3\gamma \cdot \frac{\delta R}{R_0}$$

This Equation has the form of an Wave/Oscillation

$$\frac{d^2}{d\tau^2}(\delta R) = -(3\gamma - 4) \cdot \frac{G \cdot M}{R_0^{\;3}} \cdot \delta R \qquad M := 10^6$$

The Period, T, is $\quad T_{Cepheid} = \frac{2\pi}{\omega}$

Find the Period for this simple harmonic oscillation:

Mass, M, and Radius, R, of Sun

$$M_{\odot} := 1.989 \cdot 10^{30} kg \qquad R_{\odot} := 6.96 \cdot 10^{8} m \qquad \gamma := \frac{5}{3}$$

$$\delta R(\tau) = A \cdot sin(\omega \cdot \tau) \qquad \omega^2 = (3\gamma - 4) \cdot \frac{G \cdot M}{R_0^{\;3}}$$

For a Cepheid 10X Mass & 30X Radius of Sun

$$T_{Cepheid} := \frac{2\pi}{\sqrt{(3\gamma - 4) \cdot \frac{G \cdot 10 M_{\odot}}{\left(30 R_{\odot}\right)^3}}}$$

$$T_{Cepheid} = 6.024 \cdot day$$

Modeling the Dynamics of a Cepheid: Solve for δ Radii, Velocity, and Pressure

Newton's Second Law **Use the Greek letter *thau* τ to represent the symbol for time (t)**

$$m\cdot\frac{d^2}{d\tau^2}R = \frac{-G\cdot M\cdot m}{R^2} + 4\pi R^2\cdot P \qquad P_0\cdot V_0^{\gamma} = P\cdot V^{\gamma} \qquad V = \frac{4}{3}\pi\cdot R^3 \qquad P = P_0\cdot\left(\frac{R_0}{R}\right)^{3\gamma} \qquad R = R_0\cdot r$$

$$3\cdot\gamma = 5 \qquad P = P_0\cdot\left(\frac{1}{r}\right)^5 \qquad m\cdot R_0\cdot\frac{d^2}{d\tau^2}r = \frac{-G\cdot M\cdot m}{R_0^2\cdot r^2} + 4\pi R_0^2\cdot r^2\cdot P \qquad \frac{d^2}{d\tau^2}r = \frac{-G\cdot M}{R_0^3\cdot r^2} + \frac{4\pi R_0^2\cdot r^2\cdot P}{m}$$

$$P_0 := 56\cdot kPa$$

$$\frac{d^2}{d\tau^2}r + \frac{G\cdot M}{R_0^3\cdot r^2} - \frac{4\pi R_0^2\cdot P}{m\cdot r^3} = 0 \qquad \alpha_0 := \frac{G\cdot 10M_{\odot}}{(30R_{\odot})^3} \qquad \beta_0 := \frac{4\pi(30\cdot R_{\odot})^2\cdot P_0}{10M_{\odot}}$$

$$\frac{d^2}{d\tau^2}r + \frac{\alpha}{r(\tau)^2} - \frac{\beta}{r(\tau)^3} = 0 \qquad \alpha_0 = 1.457\times 10^{-10}\,\frac{1}{s^2} \qquad \beta_0 = 1.542\times 10^{-8}\cdot\frac{km}{s^2}$$

$$\alpha := 1.4\cdot 10^{-10} \qquad \beta := 1.2\cdot 10^{-10}$$

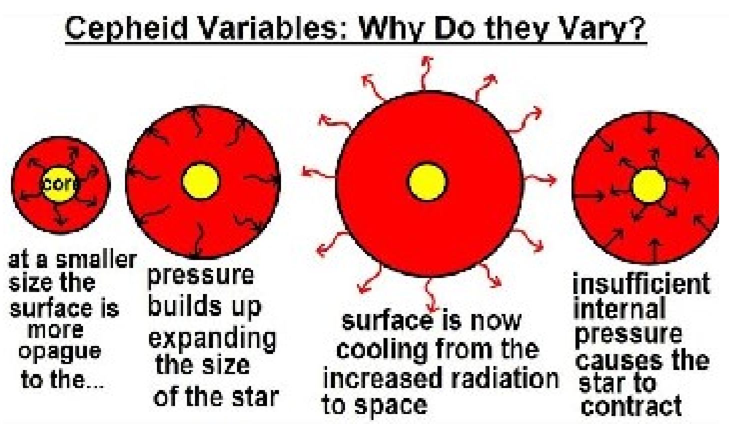
Solve Differential Equation for Cepheid Oscillations

Mathcad ODE Solver Program

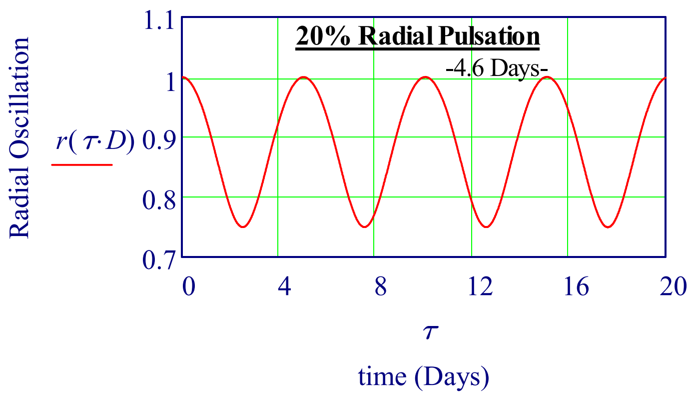
$$\text{Given} \qquad r''(\tau) + \frac{\alpha}{r(\tau)^2} - \frac{\beta}{r(\tau)^3} = 0 \qquad r(0) = 1 \qquad r'(0) = 0 \qquad r := Odesolve(\tau, 10000000)$$

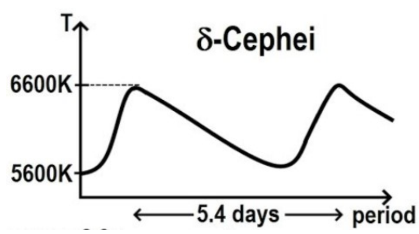
$$Day := 24\cdot 3600 \qquad D := Day$$

Below are Plots for Solution of Radius, Vel, Pressure



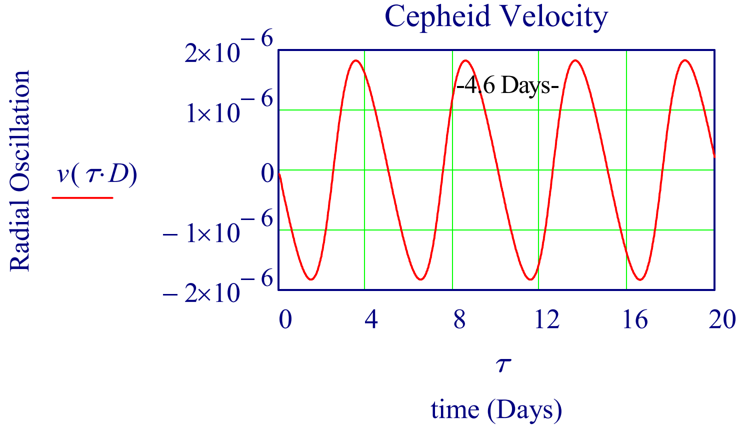
Cepheid Radial Oscillation



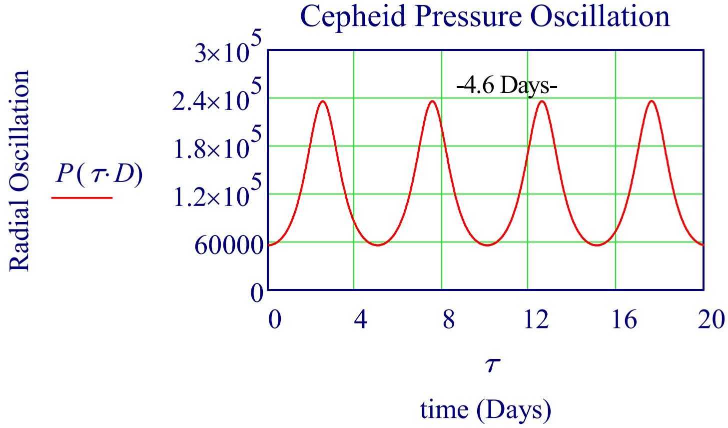


$$v(\tau) := \frac{d}{d\tau}r(\tau) \qquad P_{init} := 5.6\cdot 10^4 \qquad P(\tau) := \frac{P_{init}}{r(\tau)^5}$$

Cepheid Velocity



Cepheid Pressure Oscillation



Calibrating Cepheid period-luminosity relation from the infrared surface brightness

Astronomy & Astrophysics 534, A95 (2011)   https://www.aanda.org/articles/aa/pdf/2011/10/aa17154-11.pdf

The Cepheid period-luminosity (P-L) Relation is fundamental to the calibration of the extra-galactic distance scale and thus to the determination of the Hubble constant.

DATA: Distances & absolute magnitudes Large Magellanic Clouds (LMC) Cepheids calculated using precepts

ID#	log(P)	d	σ(d)	$(m-M)_0$	σ(m−M)	M_V	M_I	M_J	M_H	M_K	W_{VI}	W_{JK}	E(B−V)	Δφ	Δ(m−M)
		(kpc)	(kpc)	(mag)	(mag)	(mag)	(mag)	(mag)	(mag)	(mag)	(mag)	(mag)	(mag)		(mag)

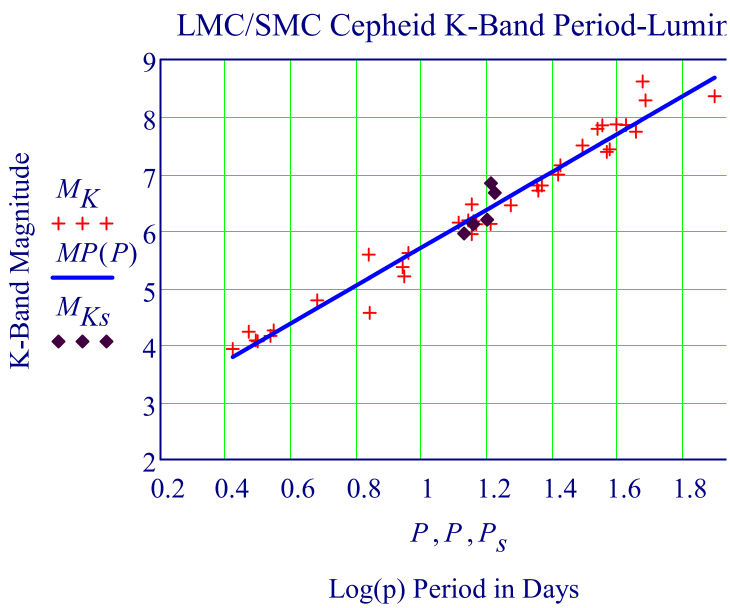
Read In Cepheid Data from File: $CPL := READPRN($"Distances and absolute magnitudes for the LMC Cepheids.tx

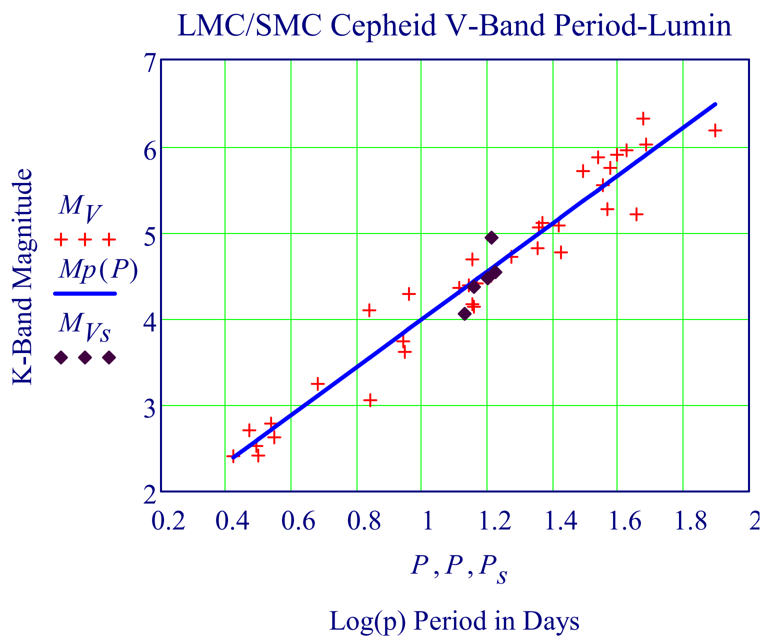
$P := CPL^{\langle 1 \rangle}$ $M_K := -CPL^{\langle 10 \rangle}$ $M_V := -CPL^{\langle 6 \rangle}$ $ab := line(P, M_K)$ $MP(p) := ab_1 \cdot p + ab_0$

$AB := line(P, M_V)$ $Mp(p) := AB_1 \cdot p + AB_0$ $ab_0 = 2.401$ $ab_1 = 3.315$ $AB_0 = 1.225$ $AB_1 = 2.774$

Read Small Megellanic Clouds: $CPL_s := READPRN($"Distances and absolute magnitudes for the SMC Cepheids.tx

$P_s := CPL_s^{\langle 1 \rangle}$ $M_{Ks} := -CPL_s^{\langle 10 \rangle}$ $M_{Vs} := -CPL_s^{\langle 6 \rangle}$





Calibrating Cepheid period-luminosity relation, **Conclusion- J. Storm, W. Gieren, P. Fouqué:**

The emerging conclusion based on our data and analysis is that for accurate distance measurements to galaxies the **K-band Cepheid Period-Luminosity** is the best suited tool: it is metallicity-independent both regarding the slope and the zero point, it is very insensitive to reddening, and it has a smaller intrinsic dispersion than any optical PL relation.

Apparent Brightness

Describe how bright a star seems as seen from Earth by its apparent brightness. This is often called the **intensity** of the starlight. Sometimes it is called the **flux of light**. The apparent brightness is how much energy is coming from the star per square meter per second, as measured on Earth. The units are watts per square meter (W/m_2).

- the distance d to the star,
- the apparent brightness b of the star, and
- the luminosity L of the star.
- All of the energy produced by the star per second must cross a sphere of radius d.
- The study of geometry tells us that area of this sphere is $4\pi d^2$

$$b = \frac{L}{4\pi d^2}$$

$$L = \left(4\pi d^2\right) b$$

Cosmic Distance Scale Summary

- Local measurements of the H_0 are now good to ~5%, and may be improved in the future
- Concept of distance ladder; many uncertainties & calibration problems, model dependence, etc
- Cepheids as the key local distance indicator
- SNe as a bridge to the far-field measurements
- Far-field measurements (SZ effect, lensing, CMB)
- Ages of oldest stars (globular clusters), white dwarfs, heavy elements consistent with CMB age
- CMB provides more precise determinations of the H 0 and other cosmological parameters.
- However, persistent discrepancy between the CMB based & Cepheid based measurements.

This may be a sign of a new physics.

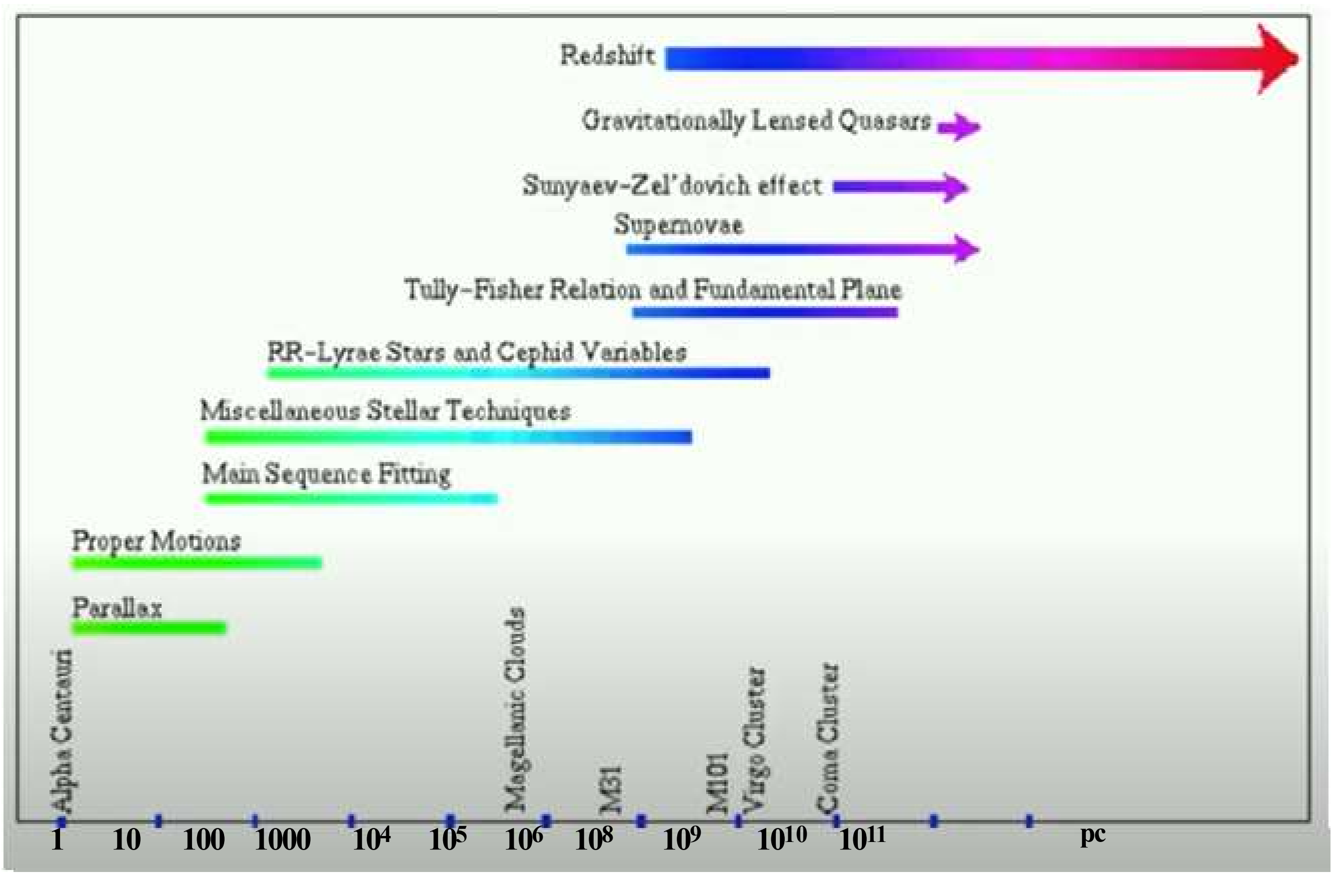
Distance Ladder

Methods yielding absolute distances:

Parallax (trigonometric. secular. and statistical)
The moving cluster method - has some assumptions
Baade-Wesselink method for pulsating stars
Expanding photosphere method for Type ll SNe Mfidel
Sunyaev-Zeldovich effect <== Model dependent!
Gravitational lens time delays <==Model dependent!

Secondary distance indicators:"**standard candle**s, requiring a calibration from an absolute method applied to local objects - **the distance ladder**:

Pulsating variables: Cepheids. RR Lyrae. Miras
Main sequence titling to star clusters
Brightest red giants
Planetary nebula luminosity function
Globular cluster luminosity function
Surface brightness fluctuations
Tully-Fisher, D_a - σ, FP scaling relations for galaxies



XXIII. 1929 Hubble's Original Observations of Galaxy Recession & Hubble Constant Calculation

The relationship between the expansion of the universe & the distance, H_0, was discovered by Edwin Hubble in 1929 from astronomical observations of Cepheid Variables, and is known as Hubble's Law. Hubble estimated velocity from redshift, z, where He assumed that z = v/c. The distance, d, is measured from parallax or a luminosity of a standard candle. Then $\mathbf{v = H_0 * r}$. Hubble thought that the redshift, z, was from the Doppler effect, v/c. He estimated the value of H_0 as 500 km/s per Mpc. Which is grossly in error because he underestimated the distance to the galaxies. The large number from the redshift velocity divided by a too small distance. Note: H = r/v. Therefore H is the reciprocal of time from expansion.

$$H_{HubbleData} := READPRN(\text{"Hubble Dataset.txt"})$$

Distance Data (Mpc)

$$d_{recH} := H_{HubbleData}^{\langle 0 \rangle}$$

$$ab4 := line\left(d_{recH}, v_{recH}\right)$$

$$H_{Hubble} := ab4_1$$

Recessional Velocity (km/s) Data from Redshift, r

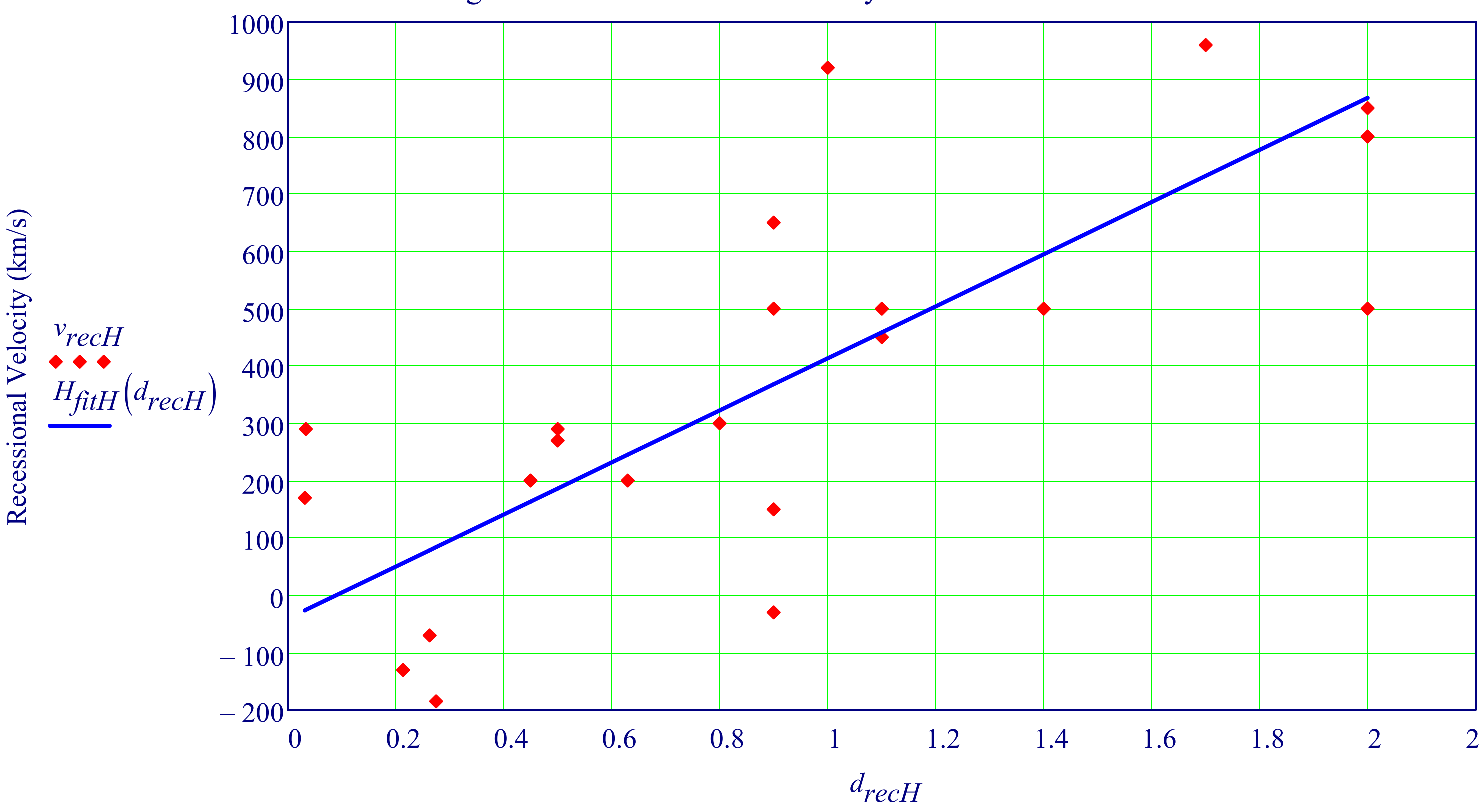
$$v_{recH} := H_{HubbleData}^{\langle 1 \rangle}$$

$$H_{fitH}(d) := ab4_0 + ab4_1 \cdot d$$

$$H_{Hubble} = 500 \frac{km}{s} \cdot Mpc^{-1}$$

$\mathbf{H_{Hubble}}$: Based on Early Measurements of Distances by Hubble. The original value of H had **Considerable Error.** Galaxies were further away than Hubble's Estimates.

Hubble's Original 1929 Recessional Velocity vs Distance: Calculation of Hubble Constant



Hubble's Original Distance to Galaxy (Mpc) Measurements from Cepheid Variables

XIV. Current Value of Hubble's Law, H0. Data: NASA Galaxy Recession from 3645 Galaxies

Hubble's original estimate estimate from Cepheids was in error. The current value is H_0 is 73 ± 1 km/sec/Mega parsec.

Standard Candle #2: Type Ia supernova For example, **all observations seem to indicate that Type Ia supernovae that are of known distance have the same brightness** (corrected by the shape of the light curve); however, the possibility that the distant Type Ia supernovae have different properties than nearby Type Ia supernovae exists. The use of Type Ia supernovae is crucial in determining the correct cosmological model. If indeed the properties of the Type Ia's are different at large distances, i.e. if the extrapolation of their calibration to arbitrary distances is not valid, ignoring this variation can dangerously bias the reconstruction of the cosmological parameters.

$parsec := 3\cdot 10^{13}\cdot km$ $\qquad Mpc := 3\cdot 10^{19} km$ $\qquad v = H_0\cdot r$ $\qquad H_0 := 73\,\frac{km}{s}\cdot(Mpc)^{-1}$

NASA/IPAC EXTRAGALACTIC DATABASE of Type IA Supernova (3645 Distance Measurements)

Read Data for 3,716 distances to 1,210 galaxies with v < 1/8 c

https://ned.ipac.caltech.edu/level5/NED1D/ned1d.html

$H_{NASA} := READPRN(\text{"Galaxy NED-1D d \& v Only.txt"})$

Number of Data Points

$rows(H_{NASA}) = 3645$

Galaxy Luminal Distance (Mpc)

$d_{rec} := H_{NASA}^{\langle 0\rangle}$

Recessional Velocity (km/s)

$v_{rec} := H_{NASA}^{\langle 1\rangle}$

Redshift z

$z := \frac{v_{rec}\cdot km}{c\cdot s}$

Corrected for Redshift

$d_{recz} := \overrightarrow{\frac{d_{rec}}{1+z}}$

Current Estimate of Hubble's Constant: Find Slope of Recessional Velocity (km/s) to Corrected Distance (Mpc)

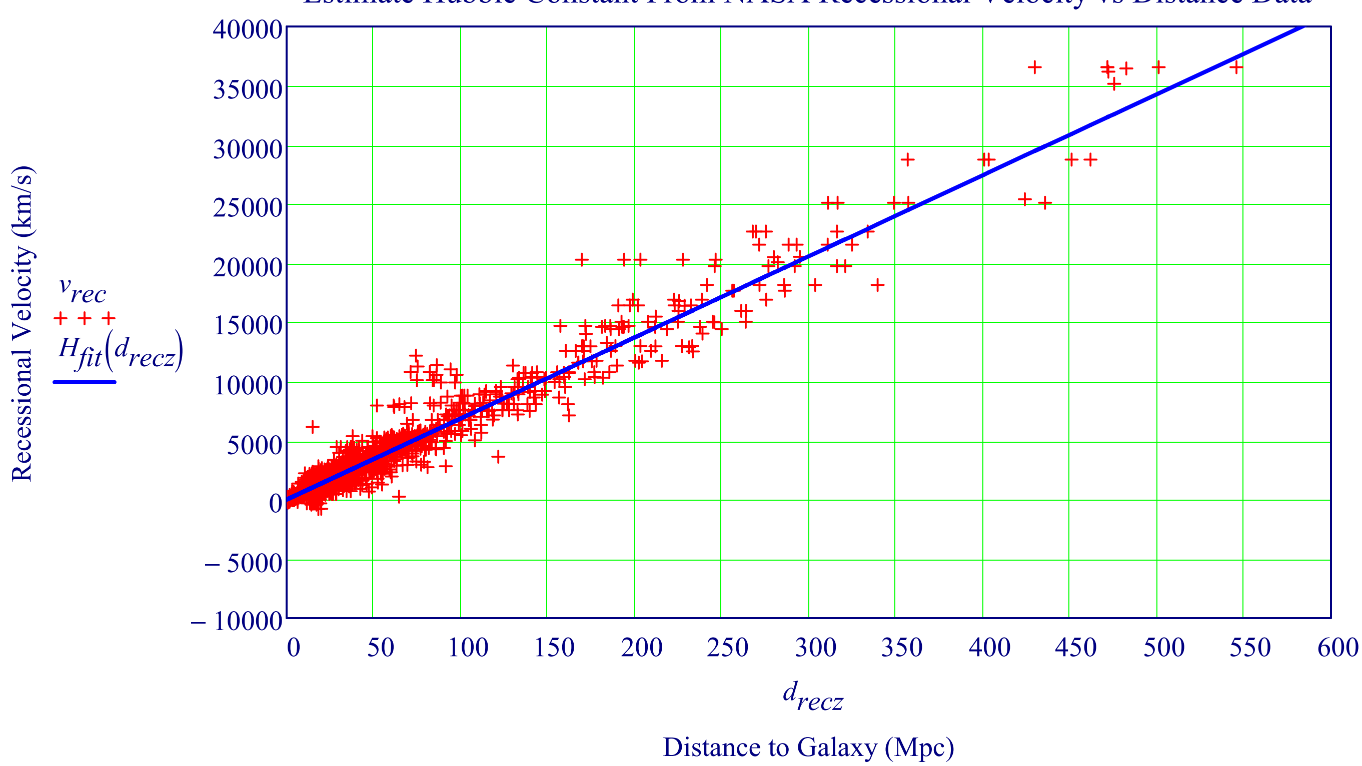
Fit Line to Data: $ab := line(d_{recz}, v_{rec})$ $\qquad H_{fit}(d) := ab_0 + ab_1\cdot d$

$H_{twk} := ab_1\cdot\frac{km}{s}\cdot Mpc^{-1}$

Calculated H_{twk} within less than a 2% Error.

$H_{twk} = 68.547\cdot\frac{km}{s}\cdot Mpc^{-1}$

Estimate Hubble Constant From NASA Recessional Velocity vs Distance Data



XV. Standard Candle 2: Hubble Space Telescope Light Curves Of Type 1a SN

Supernova Cosmology Project

"Amanullah et al. (The Supernova Cosmology Project), Ap.J., 2010

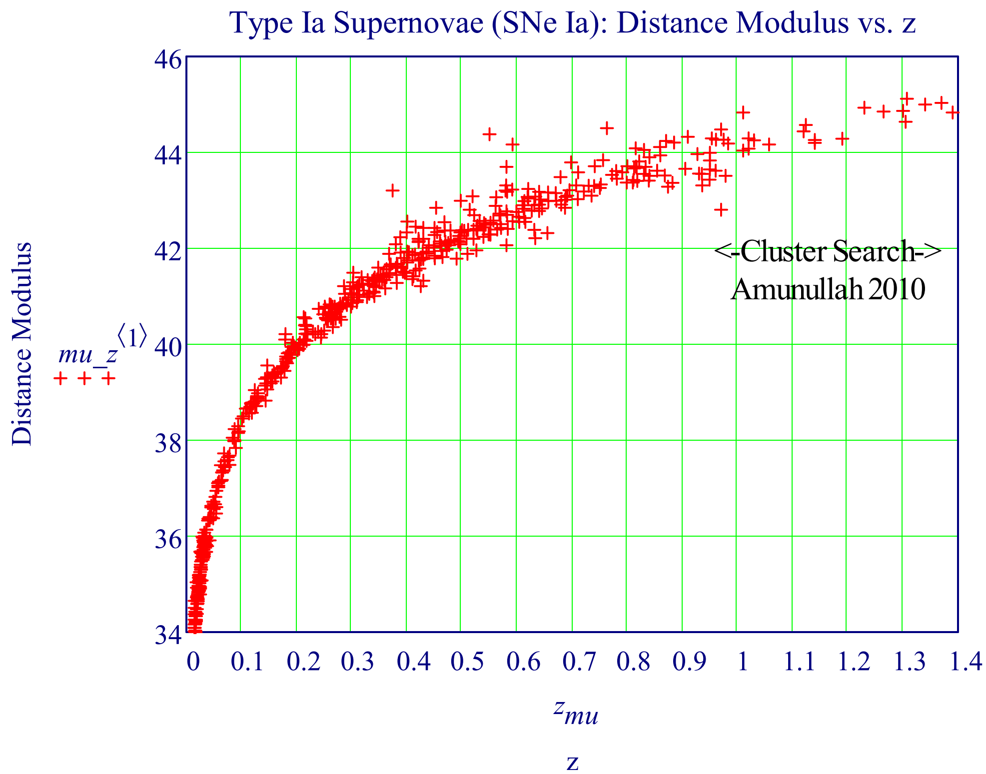
https://supernova.lbl.gov/Union/figures/SCPUnion2_mu_vs_z.txt

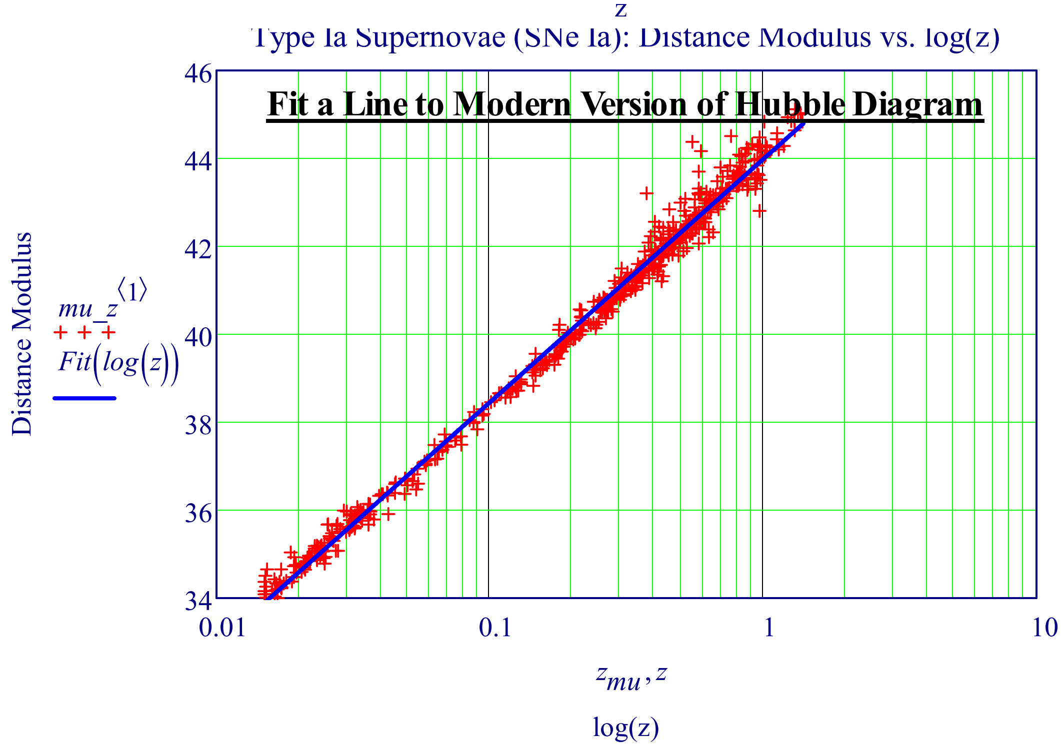
$mu\_z := READPRN(\text{"mu\_vs\_z - No Name No OL.txt"})$ $\quad mu\_z := csort(mu\_z, 0)$ $\quad z_{mu} := mu\_z^{\langle 0 \rangle}$

Fit Line to Data: $\quad \chi := line\left(log\left(z_{mu}\right), mu\_z^{\langle 1 \rangle}\right)$ $\quad Fit(z) := \chi_0 + \chi_1 \cdot z$

Modern Version of the SN Hubble Diagram

The solid line represents the best fitted cosmology for a flat Universe including the CMB and BAO constraints.





XVI. Using Gravitational Waves to Find Hubble's Constant, Hg

The gravitational wave signal emitted by the merger of two compact objects can be used as a self-calibrating standard candle. Unlike the methods to Measure the Hubble Constant, $H_{0,}$ in the followings Section X, the LIGO measurement does not use a "distance ladder". By detecting gravitational waves from merging binary neutron stars or black holes, LIGO can provide a measurement of the distance to the source and the rate at which it is movingaway from us. There are now operational detectors at LIGO Hanford and LIGO Livingston in the USA, Virgo in Italy, and KAGRA in Japan. The detectors measure the strain amplitude of a gravitational wave by using laser inferometry to detect the minuscule changes in the length of perpendicular beams as a wave passes by. The purpose of the two sites in the USA is to later out local seismic vibrations. The wave amplitude is related to the chirp mass M_c which is in turn derivable from the waveform calculated for a merger. A implied form of the relevant equations are:

For Definitions of Parameters See Sections V, XII, XXIC, and XXID.

Compare the Theoretical Magnitude-Redshift to Perlmutter 1999 SB 1A

Given the Luminosity Red Shift Relation (for k > 0):

$$D_L(z) = \frac{c(1+z)}{H_0\sqrt{\Omega_K}} \sinh\left[\sqrt{\Omega_K}\int_0^z \frac{H_0}{H(z')}\,dz'\right]$$

$$m_{bol}(z, \Omega_m) := 5\,log(1+z) + 5\,log(\chi_{em}(z, \Omega_m)) + 24$$

LIGO Parameters

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} = \frac{1}{G}\left[\frac{5}{96}\pi^{-8/3} f^{-11/3} \dot{f}\right]^{3/5}$$

$$\mathcal{M}_z = (1+z_{\rm obs})\mathcal{M}$$

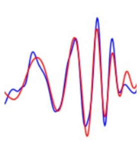
$$h(t) = \frac{\mathcal{M}_z^{5/3} f(t)^{2/3}}{d_L} F(\theta, i) \cos\Phi(t)$$

where the Luminosity Distance, $D_L(z)$ is given as the red shift integral of the Hubble parameter H(z), and the Hubble constant H_0. f is the frequency, m_1 and m_2 the merging masses, Φ(t) the phase, and Rh(t) the measured dimensionless strain of the strongest harmonic (Abbott et al. 2016). The rest-frame chirp mass is red shifted by zcobs , and F is a function of the angle between the sky position of the source and detector arms, and the inclination I between the binary orbital plane and line of sight.

The LIGO-Virgo detector network had a detection horizon of ~ 190 Mpc for binary neutron star (BNS) events (Abbott et al. 2017a), For example, the counterpart associated with GW170817 had brightness ~ 17 mag in the I band at 40 Mpc

When a binary neutron star (BNS) system merges, there is an **accompanying burst of light from matter outside** the combined event horizon. For this reason, it is known as a "**bright siren**". If the ash can be observed, the host galaxy is identified and one can use its redshift in the above equation.

The **event GW170817** was just such a BNS merger. Given the search region, an optical counterpart was found in NGC 4993 at a distance, d_L, of ~ 40 Mpc. Around f_c = 3000 cycles of the wave resolved the chirp mass in the detector frame as $M_c = 1.197\,M_\odot$ to accuracy of 1 part in 10^3, consistent with a BNS merger. The main remaining uncertainty is then the inclination angle I.



$$f_c = 3000Hz$$
$$M_z = 1.197M_\odot$$
$$d_L = 43.8Mpc$$

$$H_g = H_g(M_z, f, d_L, F, \Phi)$$

This Gives: $\boxed{H_g = 70\frac{km}{s}\cdot Mpc^{-1}}$

Abbott, B. P., et al. 2017a, PRL, 119, doi:10.1103/PhysRevLett.119.161101
—. 2017b, ApJL, 848, doi:10.3847/2041-8213/aa920c
—. 2017c, ApJL, 848, doi:10.3847/2041-8213/aa91c9

MEASURING THE EXPANSION OF THE UNIVERSE WITH GRAVITATIONAL WAVES
https://www.ligo.org/science/Publication-GW170817Hubble/flyer.pdf

XVII. Estimate of Age of Our Universe from Estimate of Hubble's Constant

Imagine the Hubble expansion scenario playing like a movie in reverse. Instead of galaxies moving away from each other as time goes forward, galaxies would rush toward each other as time goes backward. Galaxies would be closer and closer together in the past, until at some time in the distant past the matter that makes up the galaxies would have been very close together. We can extrapolate back to this time, the beginning of the Universe. If we know the expansion rate for the Universe and assume that it has been constant, we can run the clock backwards and calculate how much time the Universe has been stretching.
The age of the universe is largely determined by the rate at which it expands, and the current value of the Hubble 'constant' fixes the Hubble time. The Hubble constant is an example of a stretching rate. The Hubble constant is generally expressed in units of km/s/Mpc due to how it is measured. However, both km and Mpc are units of distance and cancel out, so the Hubble constant, or any stretching rate, actually has units of 1/time. Again, assuming that the expansion rate has been constant, **we therefore have an expression for the age of Our Universe.**

$$t_{twk\_age\_universe} := \frac{1}{H_{twk}} \qquad billion := 10^9 \qquad t_{twk\_age\_universe} = 13.869 \cdot billion \cdot yr$$

ESTIMATE AGE OF UNIVERSE FROM GALACTIC MEASUREMENTS:
13.8 Billion Years

Estimate of Radius of Curvature of the Universe: (Einstein's Old Static Idealized Model)

Putting $\dot{a} = \ddot{a} = 0$ into the Friedmann Equation, gives the radius of curvature of space in the universe

$$light\_year := c \cdot yr \qquad R_E := \frac{c}{\sqrt{4\pi \cdot G \cdot \rho_c}} \qquad R_E = 11 \cdot light\_year \cdot billion$$

Estimate the Lifetime of the Sun

Calculation is Based on the Intensity of Light from Sun and the Amount of Liberated Fusion Energy

Power to Earth From Sun: $P_{sun\_earth} := 1357 W \cdot m^{-2}$ Total Area of Earth is 4π*d^2

Rate Sun is Burning Energy = Sun's Luminosity: $L_{sun} := P_{sun\_earth} \cdot 4 \cdot \pi \cdot \left(92.027 \cdot 10^6 mile\right)^2$ $\quad L_{sun} = 3.74 \times 10^{26} \cdot W$

What Percent of Mass in Converted: One He atom has less than Mass of 4 H atoms

Particle	Proton	Neutron	2 Protons+2 Neutrons	Alpha	Difference
Units 10^{-27} kg:	1.672621637	1.674927211	6.695097696	6.64465620	0.050441496

$4 \cdot H = He + Energ$ $\quad M_{4p} := 6.692 \cdot 10^{-27} kg \qquad M_{He} := 6.644 \cdot 10^{-27} kg$

$$M_{lost\_Percent} := \left(M_{4p} - M_{He}\right) \cdot M_{4p}^{-1} \qquad M_{lost\_Percent} = 0.717 \cdot \%$$

Estimate Sun's Lifetime: Life Time = Total Energy (Esun) to Burn/fuse = Esun / Burn Rate

Only 10% of the mass of the sun is at the core where it is hot enough for fusion to occur

Mass of Sun: $M_{\odot} := 1.989 \cdot 10^{30} kg \qquad E_{sun} := 10\% \cdot M_{\odot} \cdot c^2 \cdot M_{lost\_Percent} \qquad Billion := 10^9$

$$Life_{sun} := \frac{E_{sun}}{L_{sun}} \qquad Life_{sun} = 10.878 \cdot Billion \cdot yr$$

Diverse Estimates for Mass and Densities of Matter in the Universe

Estimating the amount of baryonic matter by the number of observable stars

Estimating the amount of baryonic matter in the universe from the number of stars involves several assumptions and simplifications. Stars make up a significant portion of the visible, or baryonic, matter in the universe, but they do not account for all of it. There's also interstellar gas, planets, and other components. Here's a basic approach to such an estimate:

$$\rho_c = 0.99 \frac{kg}{m^3} \cdot 10^{-26}$$

Astronomical Estimate the Number of Stars in the Universe: Current estimates suggest that there are approximately N_{stars} stars in the observable universe. This range is based on the estimated number of galaxies in the observable universe and the average number of stars per galaxy. Number of stars in a typical galaxy (e.g. Milky

$N_{stars\_gal} := 100 \cdot 10^9$ $\quad N_{galaxies} := 2 \cdot 10^{12}$ $\quad N_{stars} := N_{stars\_gal} \cdot N_{galaxies}$ $\quad N_{stars} = 2 \times 10^{23}$

Average Mass of a Star M_{star}: The mass of stars varies widely, but for a rough estimate, you can use the mass of the Sun as an average value. $\quad M_{\odot} = 1.989 \times 10^{30}\, kg$ $\quad M_{tot\_stars} := N_{stars} \cdot M_{\odot}$

Baryonic Mass Inventory for GALAXIES and Rarefied Media
from Theory and **Observations of Rotation (RC) and Luminosity - 2023** $\quad \rho_{BaryonGal\_2023RC} := 6 \cdot 10^{-25} \frac{kg}{m^3}$

Baryonic Content of Visible Universe, Persic, 1992 $\quad \Omega_{b\ \ stars} := 0.002$ $\quad \Omega_{b\_total} := 0.003$

ρ_{Baryon} for Universe: $\quad \rho_{Baryon} := 3 \cdot 10^{-28} \frac{kg}{m^3}$

$\rho_{Baryon} \cdot \rho_c^{-1} = 0.03$

Adjust for Non-Stellar Baryonic Matter: Stars are not the only form of baryonic matter. There's also interstellar and intergalactic gas, planets, and other forms of matter. To account for this, you can adjust the total mass. Typically, the ***Mass*** of stars is estimated to be about **half** of the total baryonic matter,

$H_0 := 73 \frac{km}{s} \cdot (Mpc)^{-1}$ $\quad M_{Baryon} := 2 \cdot M_{tot\_stars}$ $\quad M_{Baryon} = 7.956 \times 10^{53}\, kg$

Estimate the Density of Matter, Mass, and Number of Atoms in the Universe

The critical density is that combination of matter and energy that brings the universe coasting to a stop at time infinity. Einstein's equations lead to the following expression for the critical density (ρ_{crit}). **A flat universe implies $\rho_{crit} = 1$**.

Equivalent to 10 Hydrogen atoms per m^3

$$\rho_c := 3 \cdot \frac{H_0^2}{8\pi \cdot G} \qquad \rho_c = 0.99 \frac{kg}{m^3} \cdot 10^{-26}$$

Estimates Based on Observable Volume of Universe Give Unreasonable Results

Radius Universe, r_{Univ} $\quad r_{univ} := 13 \cdot 10^9 \cdot light\_year = 1.231 \times 10^{26}\, m$ $\quad V_{univ} := \frac{4}{3} \pi \cdot r_{univ}^3$

Mass of Obserable Universe: $\quad Mass_{Univ} := V_{univ} \cdot \rho_c$ $\quad Mass_{Univ} = 8.274 \times 10^{52}\, kg$ $\quad V_{univ} = 7.808 \times 10^{81}\, L$

Mass Observable (Galaxies) Universe: $\quad \rho_{galax} := 3 \cdot 10^{-28} \frac{kg}{m^3}$ $\quad M_{galax} := \rho_{galax} \cdot V_{univ} = 2 \times 10^{51}\, kg$

Mass of Hydrogen:

$m_H := 1.67 \cdot 10^{-24} gm$ $\quad Number_{atoms} := \frac{Mass_{Univ}}{m_H}$ $\quad Number_{atoms} = 4.955 \times 10^{79}$

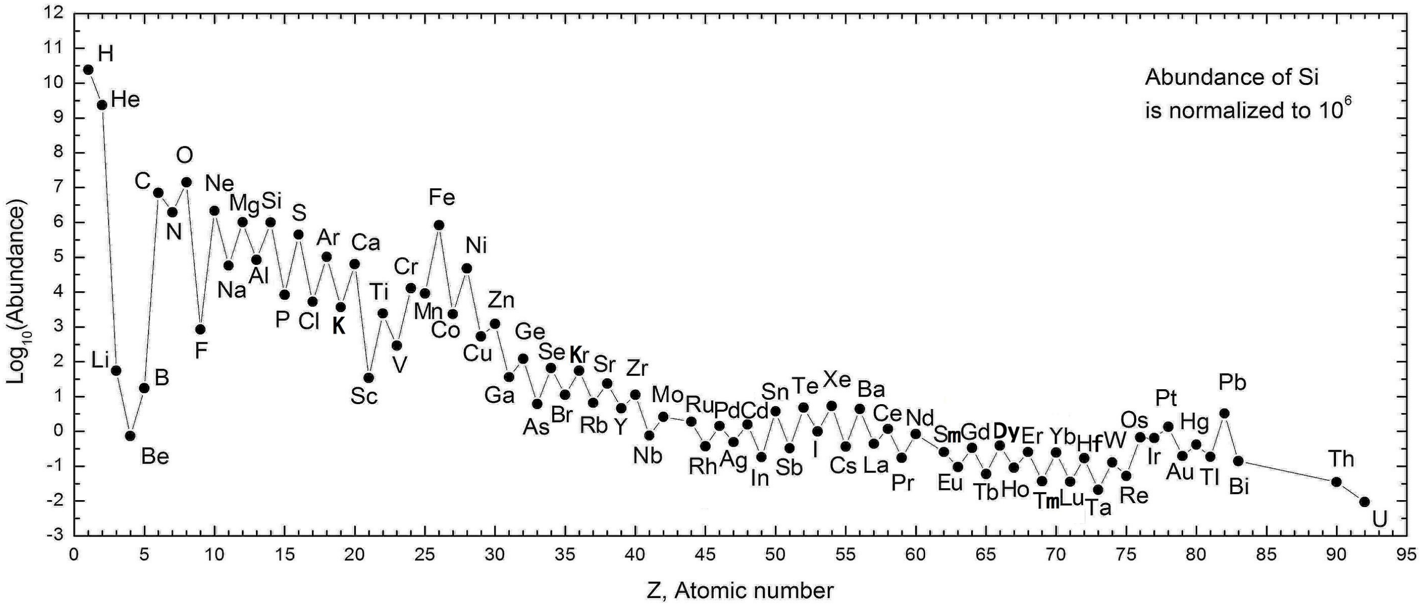
Fred Hoyle's Estimate

$M_{FH} := \frac{c^3}{2G \cdot H_0}$ $\quad M_{FH} = 8.318 \times 10^{52}\, kg$

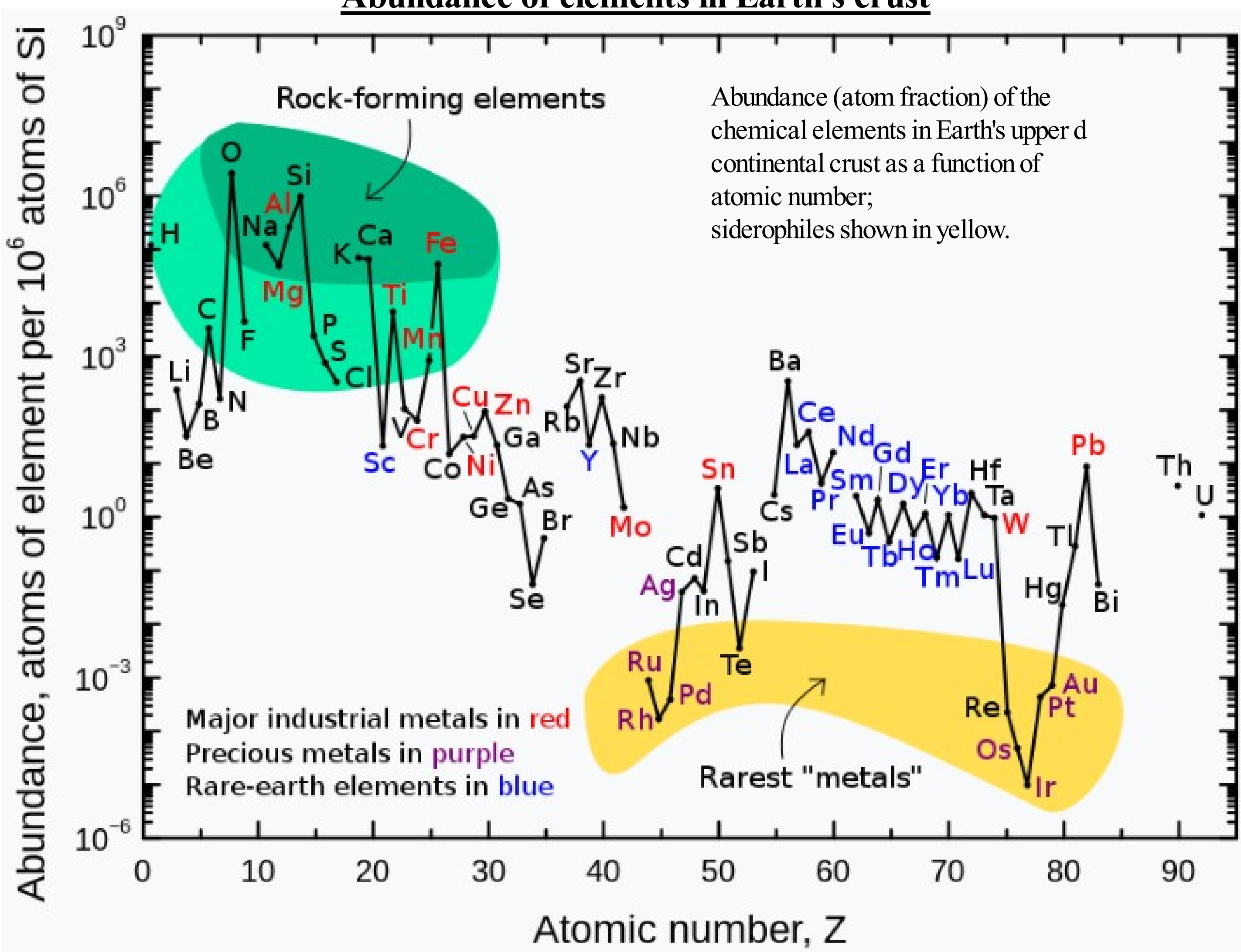
Mass from Observable Radius Fails Sanity Check

$\frac{M_{Baryon}}{Mass_{Univ}} = 9.615$ $\quad \frac{\rho_{Baryon}}{\rho_c} = 0.028$

Abundance of Elements in Solar System



Abundance of elements in Earth's crust



Graphs of abundance against atomic number can reveal patterns relating abundance to stellar nucleosynthesis and geo-chemistry. The alternation of abundance between even and odd atomic number is known as the Oddo–Harkins rule. The rarest elements in the crust are not the heaviest, but are rather the siderophile elements (iron-loving) in the Golds- chmidt classification of elements. These have been depleted by being relocated deeper into the Earth's core; their abundance in meteoroids is higher. Tellurium and selenium are concentrated as sulfides in the core and have also been depleted by preaccretional sorting in the nebula that caused them to form volatile hydrogen selenide and hydrogen telluride.

There are 92 elements. All the two of them are extremely anomalous, in terms of what we see in the crust of the earth, relative to what we see in Rocky material elsewhere in the universe. The two that are normative are manganese and iron. Everything else is anomalous, and in some cases, extremely anomalous. So for example, the crust of the earth is 630 times as much thorium 340 times as much uranium as what we see in Rocky material in the rest of the universe. And as thanks for that super abundance of uranium and thorium, our planet a long lasting hot core. And that hot liquid iron core, being circulated, has enabled our planet to have a strong magnetosphere and developing us that allows us to be protected from deadly solar and cosmic radiation. It also prevented the atmosphere and the oceans of the Earth from being sputtered away by the particle radiation from the sun.apacity. So we got 60 times less sulfur, that's what enables us to grow food, you're not going to grow any food or crops on Mars, because there's way too much sulfur there. But you can on the earth, so we're deficient by a factor of 60 times in sulfur. But were abundant by a factor of 60 times in aluminum, 90 times in titanium, which enables us to construct aircraft that can fly all over the world. These are light metals that have very high strength. And so we have in a very anomalous high abundance of these valuable elements. And they're 22 elements we see in the periodic table, that are what we call vital poisons. If they exist in the crust of the earth, at too high of an abundance level, it'll kill us, but too low of an abundant level, it will also kill us.

So we have to have just the right amount of molybdenum, and the crust of the earth, just the right amount of iron, just the right amount of arsenic. There's actually proteins in your body that need arsenic, but you only need a very, very tiny amount, and you get above that tiny amount, the arsenic will kill you. And it has to be at just the right level. And so all 22 of these vital poisons are extremely anomalous, and their abundance level here on planet Earth. And we don't see it anywhere else in the universe. So it really does look like somebody engineered it to get it just right. And astronomers again have discovered how this happened. How the early solar system formed in a gigantic cluster of about 20,000 stars that existed much closer to the center of the galaxy than the solar system exists today. And in that dense cluster of stars, the early emerging solar system got exposed to three different kinds of supernova eruption events. It got exposed to neutron stars merging together to make black holes, where the supernova and neutron star merging events happen at exactly the right time, and the right distance from the earth so that the earth was not destroyed. But on the other hand, got sufficiently enriched in all these elements and sufficiently depleted and elements be a problem. And then when all that enrichment depletion was accomplished, we got kicked out of the birth cluster and driven to a distance twice as far away from the center of the galaxy, what kicked us out, it was a gravitational slingshot, where our solar system was interfacing with four or five very massive stars that slung us out of the birth cluster. And then when we got to the ideal place for advanced life, we again engage another four or five, six massive stars that halted our movement. And so we were born in the most dangerous part of our galaxy. And we ended up in the safest part of our galaxy, but only after we got in rich. Now, it's also true that our planet Earth is anomalous, compared to all the other planets, and asteroids we see in in our solar system. And that's because our Earth formed, in a way incredibly different from the other planets, the other planets formed by gravitational accretion. And our solar system began with 10 planets, not eight, five gas giants and five rocky planets. Two of those rocky planets, so proto Earth and Thea collided with one another, when the Earth had oceans 1000s of kilometers deep, that very deep ocean cushion the collision, so the earth was not destroyed. In fact, what happened, most of the mass of thea got absorbed into the earth. So the earth became bigger, more massive and denser. There is a debris cloud around the new forming Earth, that condensed to make the moon. And so we have this relatively small planet, orbited by a gigantic moon that stabilizes the tilt of a rotation axis. It ensured that at the just right time for human beings, we have a rotation rate slowed down to 24 hours. And that this gas giant planet, it got kicked out by a gravitational interaction with Jupiter and Saturn. And that gravitational interaction basically slimmed down Mars from being a planet about twice the mass of Earth, down to a planet. That was only one night the mass of the Earth. This was called the Smar small Mars problem. It took 20 years for astronomers to determine how did Mars get to be so small, but we now recognize if it wasn't for that transformation of Mars, there'd be no possibility for advanced life to exist on planet Earth.

ESTIMATIONS OF TOTAL MASS AND ENERGY OF THE OBSERVABLE UNIVERSE

Dimitar Valev, Physics International 5 (1): 15-20, 2014

To determine gravitational and kinetic energy of the observable universe, information of the size and total mass of the universe are needed. There are different estimations of the mass of the observable universe covering very large interval from 3×10^{50} kg (Hopkins, 1980) to 1.6×10^{60} kg (Nielsen, 1997). Also the estimations of the size (radius) of the universe are from 10 Glyr (Hilgevoord, 1994) to more than of 78 Glyr (Cornish et al., 2004).

Estimate Mass of Universe by Dimensional Analysis

The fundamental parameters as the gravitational constant G, speed of the light c and the Hubble constant $H \approx 70$ km s^{-1} Mp_{s-1} (Mould et al., 2000) determine the global properties of the universe. Therefore, by means of these parameters, a mass dimension quantity m_{dim} related to the universe could be constructed:

$$m_{dim} = kc^{\alpha} G^{\beta} H_0^{\gamma}$$

where, k is a dimensionless parameter of the order of magnitude of a unit and α, β and γ are unknown exponents which have been found by means of analysis. Taking into account the dimensions of the quantities in the mx Equation we obtain the system of linear equations for unknown exponents Equations:

We use the determinant Δ of the system for the above mx Equation to find the parameters by Kramer's formula.

$$\alpha + 3\beta = 0 - \alpha - 2\beta - \gamma = 0 - \beta = 1 \qquad \Delta = \begin{vmatrix} 1 & 3 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & 0 \end{vmatrix} = -1 \qquad \alpha = \frac{\Delta_1}{\Delta} = 3 \; \beta = \frac{\Delta_2}{\Delta} = -1 \; \gamma = \frac{\Delta_3}{\Delta} = -1$$

Check Exponent Values $\alpha := 3$ $\beta := -1$ $\gamma := -1$ **Compare this to the above estimate**

This result gives the correct solution for exponent α,β,λ $\alpha + 3\beta = 0$ $-\alpha - 2\beta - \gamma = 0$ $-\beta = 1$

Theoretical Estimate of the Maximum Number of Stars in Universe

Mass from Dimensional Analysis **Mass from Critical Density**, ρ_c

$$m_{dim} = kc^{\alpha} G^{\beta} H_0^{\gamma} \qquad m_{dim} := \frac{c^3}{G \cdot H_0} \qquad m_{dim} = 1.664 \times 10^{53}\, kg \qquad Mass_{Univ} = 8.274 \times 10^{52}\, kg$$

WMAP Estimate: $Pecent_{baryonic} := 0.046$

$$MassUniv_{baryonic} := Pecent_{baryonic} Mass_{Univ}$$

$$MassUniv_{baryonic} = 3.806 \times 10^{51}\, kg$$

The most common type of star turns out to be one with about 0.25 solar mass.

$$M_{typical} := 0.25 \cdot M_{\odot}$$

$$Num_{stars} := \frac{MassUniv_{baryonic}}{M_{typical}}$$

$$Num_{stars} = 7.654 \times 10^{21}$$

History of Numbering of the Stars - Cosmology

We Live in a Time of Exponential Growth in Our Knowledge of the Universe (Cosmology)

Estimate of Order of Magnitude (# of Zeros) of Number of "Known Stars"

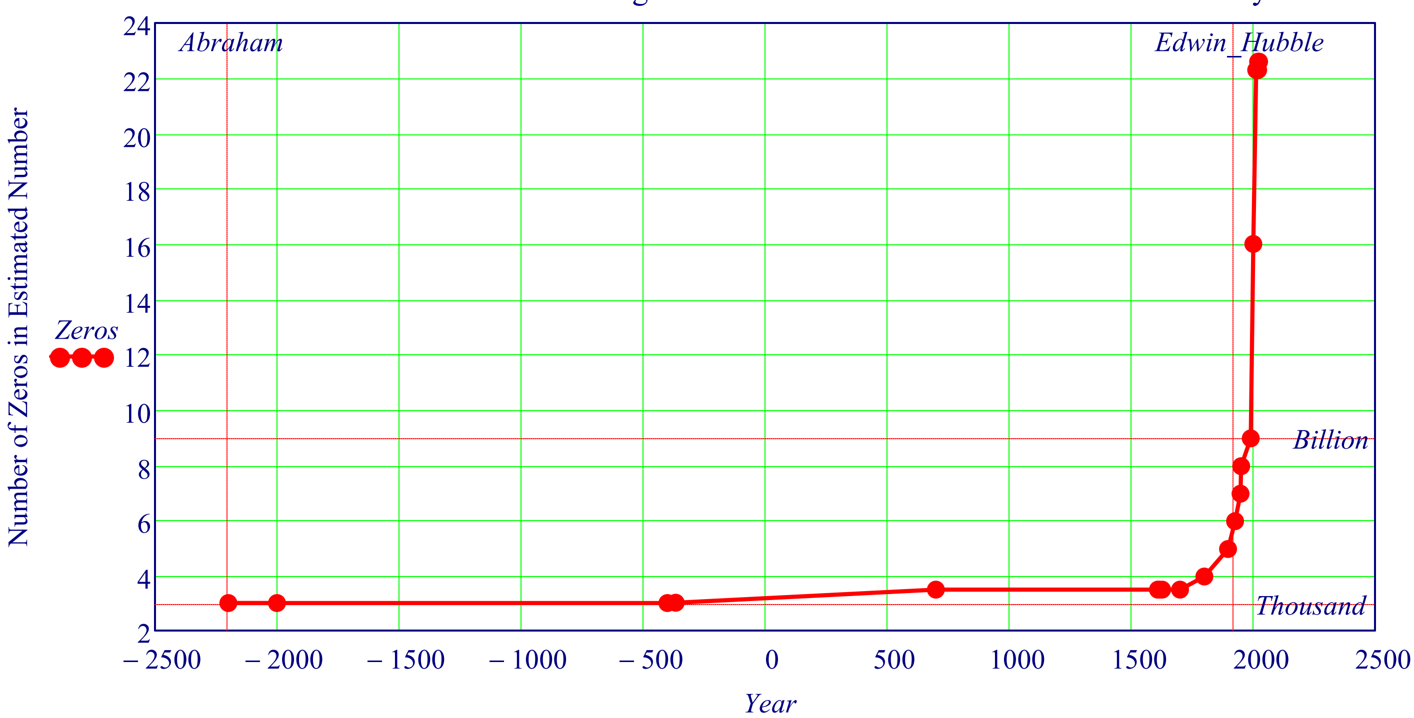
List Number of Stars that were Cataloged, Known, or Estimated Based on Observations.
Example: 2500 BC could only see about 3,000, Yerkes Observatory cataloged 13,655 stars in 1800.
We are interested only in obtaining the Order of Magnitude of the Known, Cataloged, or Estimated Stars.

$N_{stars} := READPRN(\text{"Num of Stars2.txt"})$ $\quad Abraham := -2200 \quad Edwin\_Hubble := 1925$

$Zeros := log\left(N_{stars}{}^{\langle 1 \rangle}\right) \qquad Year := N_{stars}{}^{\langle 0 \rangle} \qquad rows\left(N_{stars}\right) = 20$

$Thousand := 3 \qquad Billion := 9$

Estimates of the Order of Magnitude of Obserable Stars and Year of Discovery



Year of Discovery/Estimate

Ratio of Baryonic to Dark Matter

To calculate this ratio in a specific galaxy, astronomers **measure the rotation speed of the galaxy at various distances** from its center. They then create a rotation curve based on the visible matter (using the mass of stars, gas, etc.) and compare it with the observed rotation curve. The difference between these curves indicates the amount of dark matter. By integrating the mass profiles of both baryonic and dark matter, astronomers can estimate their respective contributions to the galaxy's total mass. While the exact ratio of dark matter to baryonic matter varies, a commonly cited average is that dark matter makes up about 85% of the total matter content in galaxies, with baryonic matter constituting about 15%. This implies a ratio of approximately 5.7:1 (dark matter to baryonic matter).

Applying this ratio gives for the total Matter in Universe $\quad Tot_{matter} := M_{Baryon} \cdot (1 + 5.7) = 5.331 \times 10^{54}\, kg$

XVIII. Uniformity of the CMBR is Evidence for Istropic Expansion and the Big Bang

1998 COBE Far Infrared Absolute Spectrophotometer Monopole Spectrum Measurements

Assess If the Origin of the Cosmic Microwave Background Radiation (CMBR) is from the Big Bang

COBE Measurements of CMBR Spectrum - Test: Surface of Last Scattering (from Clouds)? Thermal Blackbody?

Column 1 = Reciprocal Wavelength, λ, from Table 4 of Fixsen et al., in units = cm^{-1}

Column 2 = Intensity of FIRAS monopole spectrum computed as the sum of column 3, units = MJy/sr

$CMBR := READPRN(\text{"iras\_monopole\_spec\_v1.txt"})$ $\quad T_{mw} := 2.7250K$

$\lambda := CMBR^{\langle 0\rangle} \qquad \lambda_6 = 4.99 \qquad I := CMBR^{\langle 1\rangle} \qquad n := 0, 1 .. rows(I) - 1$

$k_b := 1.3806505\cdot 10^{-23}\cdot\frac{joule}{K} \qquad h := 6.6260693\cdot 10^{-34}\cdot joule\cdot sec$

Determine How Well COBE Spectrum Matches the Stretched Black Body Radiation at T = 2.750 K

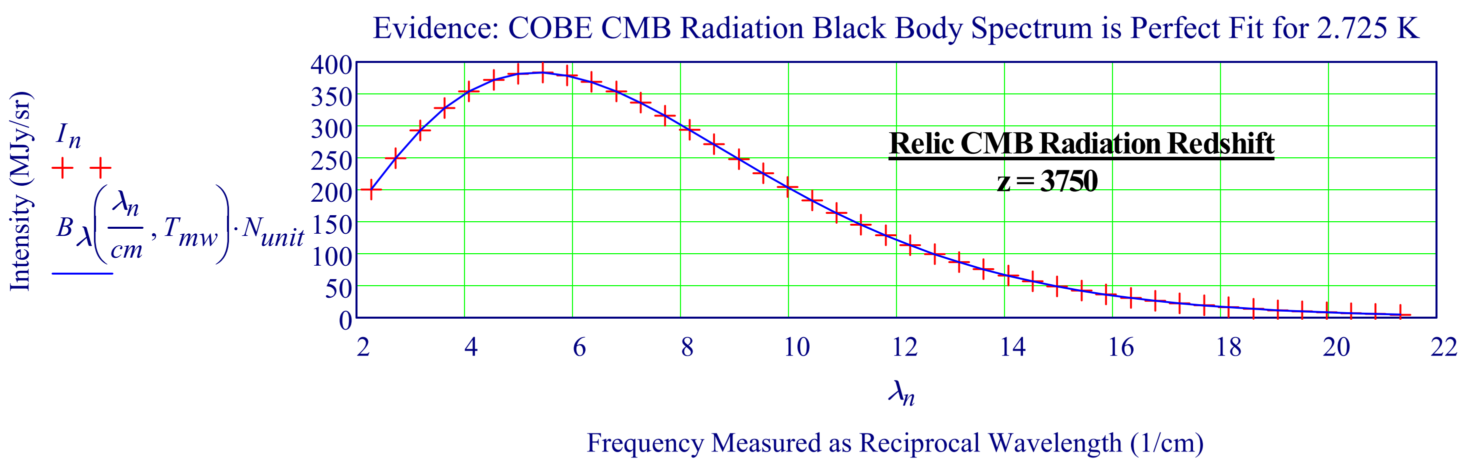
Model: Equation for Intensity of Ideal Black Body Spectrum | **Normalize Units at λ = 4.99**

$$B_\lambda(\lambda, T) := 2h\cdot c^2\cdot\lambda^3\cdot\left(e^{\frac{h\cdot c\cdot\lambda}{k_b\cdot T}} - 1\right)^{-1} \qquad N_{unit} := I_6\cdot B_\lambda\left(\frac{4.99}{cm}, T_{mw}\right)^{-1}$$

Wavelength of Light, λ, Stretches with Expansion

Stretches with the scale factor, a and λ stretch factor, z.
Given wavelength at emission, λ_o, λ today is

$$\lambda = \frac{1}{a(t)}\cdot\lambda_0 = (1 + z)\lambda_0$$



CMB Energy: $N19 := 10^{-19} \qquad eV := 1.6\cdot 10^{-19} C\cdot volt = 1.6J\cdot N19 \qquad k_b\cdot 2.75K = 2.373\times 10^{-4}\cdot eV$

Measured Uniformity (Low Error) of CMBR Temperature Reveals An Almost Perfect 2.725K Spectrum

$$Error\% := \frac{1}{rows(I)\cdot 100}\left[\sum_n \left(I_n - B_\lambda\left(\frac{\lambda_n}{cm}, T_{mw}\right)\cdot N_{unit}\right)\right] \qquad Error\% = 0.0014$$

Scaling ==> $Temp(t) = \frac{T_0}{a(t)} \qquad \lambda := \frac{c}{\nu}$ ~ **a**

CONCLUSION - ORIGIN OF CMBR:

The CMB radiation was emitted 13.7 billion years ago, only a few hundred thousand years after the Big Bang, **long before stars or galaxies ever existed**. Radiation's temperature is defined by the wavelength of the individual photons that make it up. **As the Universe expands**, not only does the radiation get **less intense**, but the **stretching of space** will stretch the wavelength of the photons from the Big Bang, which **decreases the energy** of the photons to longer wavelengths, which correspond to the energy of **lower temperatures**. When neutral atoms form, the radiation can no longer interact, and simply flies in a straight line until it interacts with something. **13.8 billion years later, that something is our eyes** and instruments, revealing an **ultra-cold, uniform bath** of radiation at 2.725 K. This is **Evidence of radiation from a hot, dense phase** in the past that many had theorized as representing the origin of our expanding Universe.

XIX. Planetary Data and Classical Newton's Calculation of Planetary Velocity

Read Planetary Data (MDD) and Compare to Calculated Velocity from Newton's Equation, vss

https://nssdc.gsfc.nasa.gov/planetary/factsheet/

$MDD := READPRN(\text{"Planets Mass Dist Density.txt"})$ $\quad MDD := MDD^T$

MERCURY VENUS EARTH MARS JUPITER SATURN URANUS NEPTUNE PLUTO

Mass Density Gravity EscapeVel Period Day Distance Perih, Aph, OrbPeriod OrbVelocity

$Mass := MDD^{\langle 0 \rangle} \quad Dist := MDD^{\langle 7 \rangle} \quad Vel_{Data} := MDD^{\langle 11 \rangle} \quad v_{Earth} := Vel_{Data_2}$

Analytic Estimate: Newton's Model Equation for Velocity vs. Distance, d

$d_{Earth} := Dist_2$

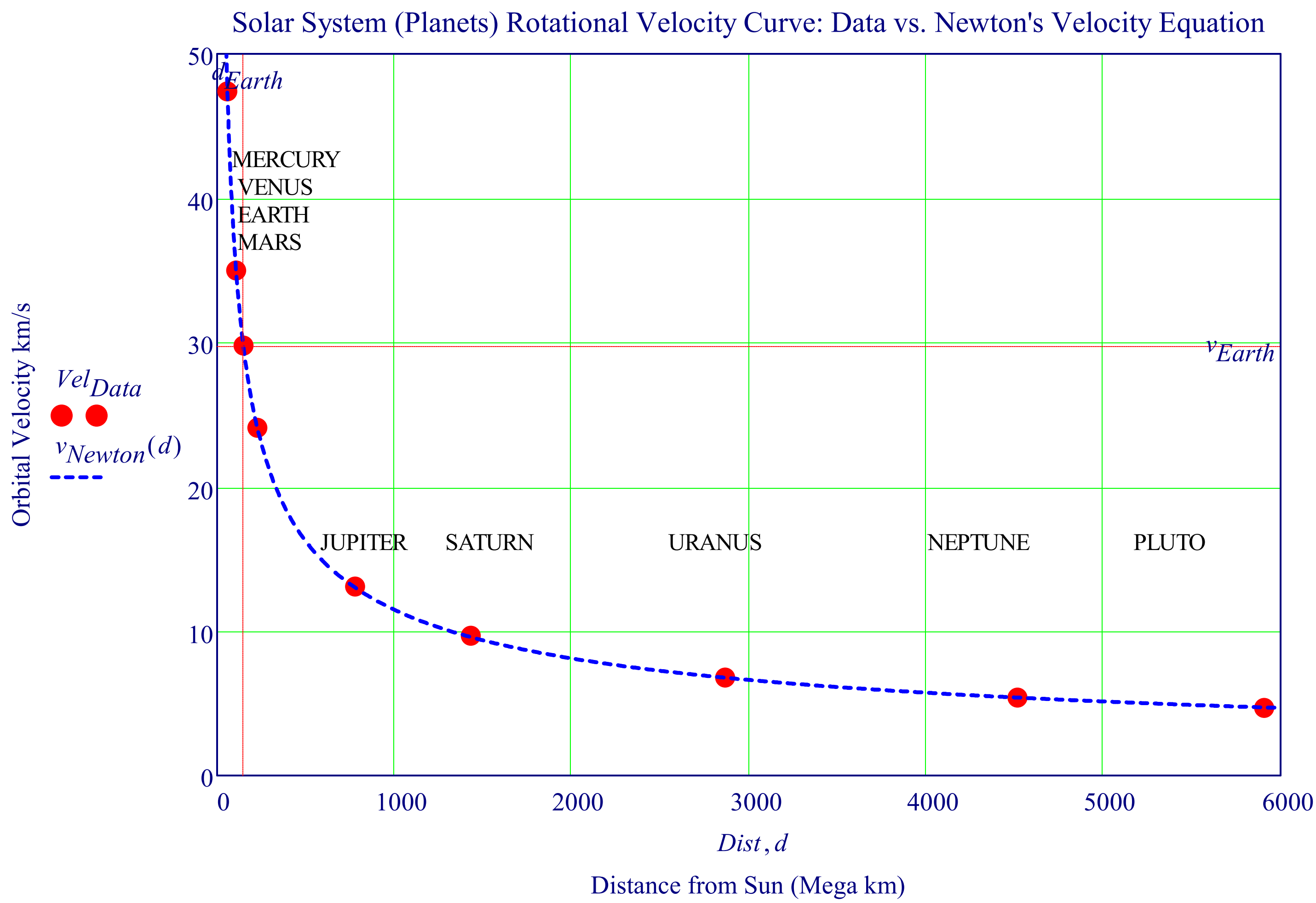
$M_{\odot} := 1.98 \cdot 10^{30} \cdot kg \quad v_{Newton}(d) := \sqrt{G \cdot \frac{M_{\odot}}{d \cdot 10^6 \cdot km}} \cdot \frac{1}{\frac{km}{s}}$

$v_{Newton}(6000) = 4.692$

Velocity vs Distance Curve, Falls Off Rapidly with Distance, is What is Expected for Galaxy Rotational Velocity

$d := 0, 10 .. 6000$

Note Excellent Agreement Between Planetary Velocity Data and Newton's Prediction



XX. Indication of Cold Dark Matter: Rotational Velocity Curves - Milky Way Galaxy

Observed Rotational Velocity of Galaxies - Velocity Does Not Falloff Rapidly Like Planets

Observing the rotational velocity of stars in galaxies is a fundamental tool to derive the mass distribution in the galaxy. Estimating the velocity of galaxy based on visible based on Classic Newton's or Kepler's Law's gives a velocity curve (VR_{Kep}) that falls off quickly with distance. The actual Galactic Velocity acts like there is a halo of matter around galaxy.

Cold Dark Matter constitutes about 26.5% of the mass–energy density of the universe. The remaining 4.9% comprises all ordinary matter observed as atoms, chemical elements, gas and plasma, the stuff of which visible planets, stars and galaxies are made. The great majority of ordinary matter in the universe is unseen, since visible stars and gas inside galaxies and clusters account for less than 10% of the ordinary matter contribution to the mass–energy density of the universe.

We want to calculate the Fraction of Cold Dark Matter in the Milky Way Galaxy

Bright Matter Mass of Milky Way Galaxy: $M_{mwg} := 6.3\cdot 10^{41}\cdot kg\cdot 0.1$ $\quad kpc := 3.08\cdot 10^{16} km$

Radial Scale Length: $R_0 := 2.1 kpc$ $\quad r_c := 16 kpc$ $\quad M_o := 6\cdot 10^{42} kg$

Expected Galactic Velocity Distribution (VKep) based on Keplerian type (Sun - Planetary) Mass Distribution

This is the type of falloff of velocity with distance we would expect to see from the mass of ordinary visible matter

$$VR_{Kep} := READPRN(\text{"Galaxy Expected.csv"}) \quad R_{Kep} := VR_{Kep}^{\langle 0\rangle}\cdot 4 \qquad X := 1 - 0.7\frac{R_{Kep}}{100}$$

$$V_{Kepler} := \overrightarrow{\left(VR_{Kep}^{\langle 1\rangle}\cdot X\right)}$$

See Graph of Galaxy Velocity on Next Page

Determination of Amount of Dark Matter from Rotation Curve (RC) of Milky Way Galaxy

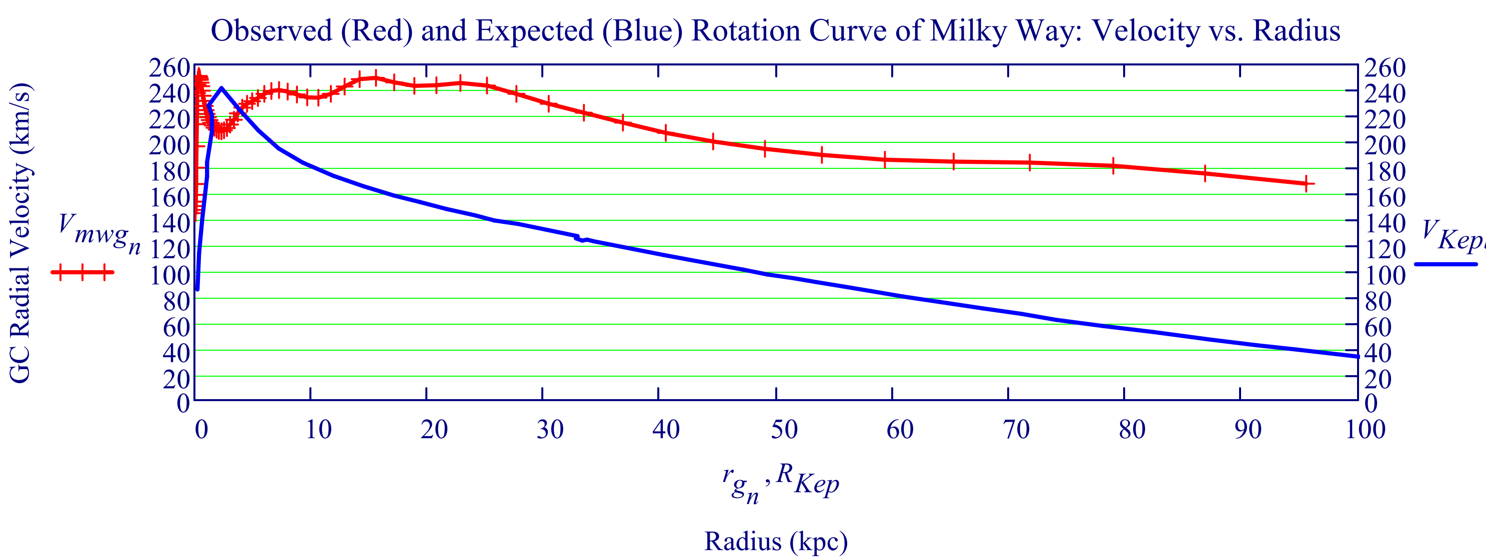
Radius (kpc), $V_{rotation}$ (kms/s), Std Dev (km/s)

DATA: Rotation Curve Parameters of the Milky Way and the Dark Matter Density, Yoshiaki Sofue, mdpi.com
Institute of Astronomy, Graduate School of Sciences, The University of Tokyo, Mitaka, Tokyo, Japan

Read Data for Rotation Curve: $RCMW := READPRN(\text{"Rotation curve of the Milky Way.txt"})$

Milky Way Velocity: $V_{mwg} := RCMW^{\langle 1\rangle}$ Let r_g be the radius of Galaxy: $r_g := RCMW^{\langle 0\rangle}$ $\quad n := 0 .. rows(RCMW) - 1$

Note the two prominent rotation velocity dips at radii 3 and 9 kpc.



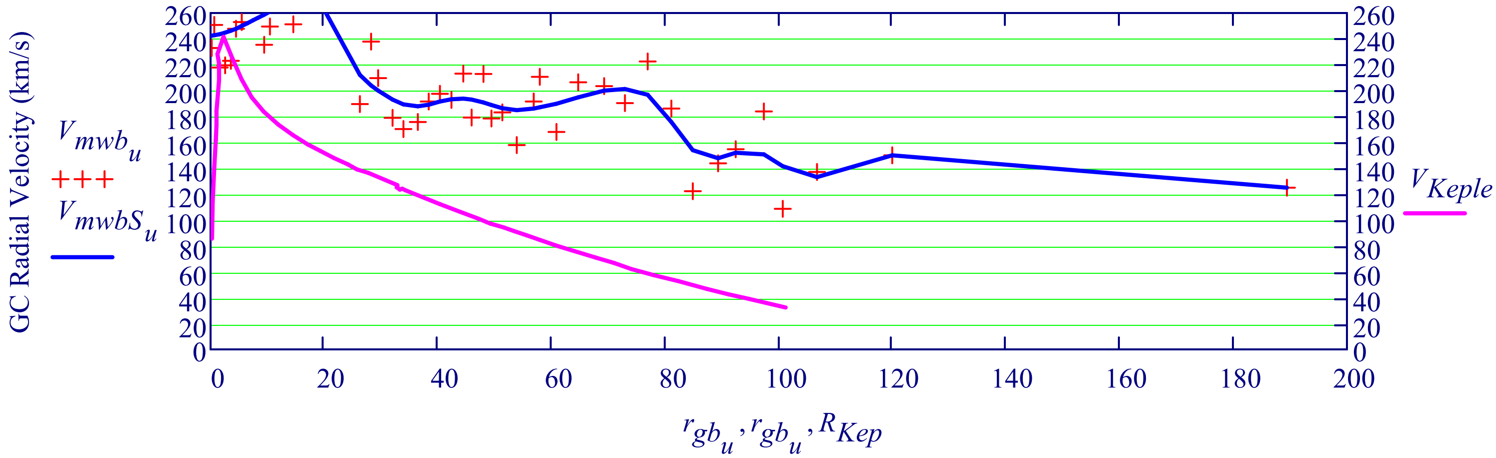
ROTATION CURVE OF THE MILKY WAY OUT TO ~200 kpc

https://iopscience.iop.org/article/10.1088/0004-637X/785/1/63/pdf

$MWB := READPRN("RC MILKY WAY 200 kpc -Bhattacharjee.txt")$

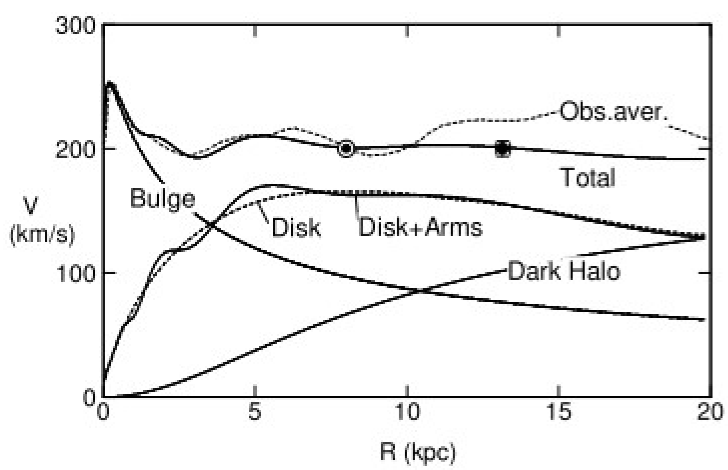
$V_{mwb} := MWB^{\langle 1 \rangle}$ $r_{gb} := MWB^{\langle 0 \rangle}$ $u := 0, 1 .. rows(MWB) - 1$ $V_{mwbS} := ksmooth\left(r_{gb}, V_{mwb}, 12\right)$

Observed (Red+) and Smoothed (Blue) Rotation Curve of MWay: Velocity vs. Radius



Composite Rotation Curve of Milky Way Galaxy Showing Mass Components

Composite Rotation Curve including the bulge, disk, spiral arms, and dark halo.
Yoshiaki Sofue, Mareki Honma , and Toshihiro Omodaka, PASJ 2018



The rotation velocity is written by the gravitational potential as $V(R) = \sqrt{R \cdot \frac{\partial}{\partial R} \Phi}$

where $\Phi = \sum_i \Phi_i$

with Φi being the potential of the i-th mass component

Knowing that Vi(R) = R ∂ Φi / ∂ R, we have $V(R) = \sqrt{\sum_i V_i^2}$

Mass Components

Below, the subscript BH represents black hole, b stands for bulge, d for disk, and h for the dark halo.The contribution from the black hole can be neglected in sufficiently high accuracy, when the dark halo is concerned.

$$V(R) = \sqrt{V_{BH}(R)^2 + V_b(R)^2 + V_d(R)^2 + V_h(R)^2}.$$

The **mass components** are usually assumed to have the following functional forms:

The GC of the **Milky Way** is known to **nest a massive black hole** of mass of MBH ~ $\mathbf{4 \times 10^6\ M_\odot}$

The RC is assumed to be expressed by a curve following the Newtonian potential of a point mass at the nucleus. and the rest of total mass is what is called dark matter---material that does not emit any light (a small fraction of it is ordinary matter that is too faint to be detected yet) but has a significant amount of gravitational influence. The total mass of the galaxy, M_g, including the extended dark halo, has been measured by analyzing the outermost RC and motions of satellite galaxies orbiting the galaxy, and the **mass up to ~100–200 kpc** has been estimated to be $\mathbf{3 \times 10^{11}\ M_\odot}$. ,

Mg = 0.3 Trillion Sun Masses

Where $M_\odot$ is the mass of Sun $\quad M_\odot := 1.989 \cdot 10^{30} kg \quad M_g := 3 \cdot 10^{11} \cdot M_\odot \quad R_g := 8kpc$

Fit a Curve, (VFit), to the Milky Way Rotation Curve

$$V_{Fit} := ksmooth\left(r_g, V_{mwg}, 10\right)$$

Simple Model for Milky Way Galaxy that Approximates Galaxy Rotation Curves

Galactic Model: *Simple Model for Explaining Galaxy Rotation Curves*, A. Wojnar, Sporea

Model Parameters: M_0 the total galaxy mass, R_0 the observed scale length of the galaxy, r_c the core radius and fitting parameters b and β

Galactic Velocity Curve Fitting Model, vmw, with Five Fitting Parameters, Mg, R0, rc, b, and β

$M_{gas} := 10^{9.68} \cdot M_\odot \quad M_s := 10^{9.76} \cdot M_\odot \quad R_0 := 2.6kpc \quad r_c := 0.88kpc \quad b := 0.352 \quad \beta := 1$

$M_g = 5.967 \times 10^{41} kg \qquad M_{tot} := M_{gas} + M_s \qquad X_M := \left(M_{gas} + M_s\right) \cdot M_g^{-1}$

$$v_{model}(r) := \sqrt{\frac{G \cdot M_{tot}}{r} \cdot \left(\sqrt{\frac{R_0}{r_c}} \cdot \frac{r}{r + r_c}\right)^{3\beta} \cdot \left[1 + b \cdot \left(1 + \frac{r}{R_0}\right)\right]} \qquad v_{model}(20kpc) = 202.81 \cdot \frac{km}{s}$$

The Dark Halo Density profile:

DM Model: *Unified Rotation Curve of the Galaxy, Decomposition Bulge, Disk, Dark Halo,* Sofue

ρ_{hc} and R_h are constants giving the central mass density (ρ_{hc}) and scale radius of the halo, respectively

$$\rho_{hc} := 0.03 \cdot M_{\odot} \cdot parsec^{-3} \qquad R_h := 5.5kpc \qquad light\_year := 0.000306kpc$$

$$\rho_{halo}(r) := \rho_{hc} \cdot \left[1 + \left(\frac{r}{R_h}\right)^2\right]^{-1} \qquad V_{inf} := \sqrt{4 \cdot \pi \cdot G \cdot \rho_{hc} \cdot R_h^{\,2}} \qquad V_{inf} = 230.557 \cdot \frac{km}{s}$$

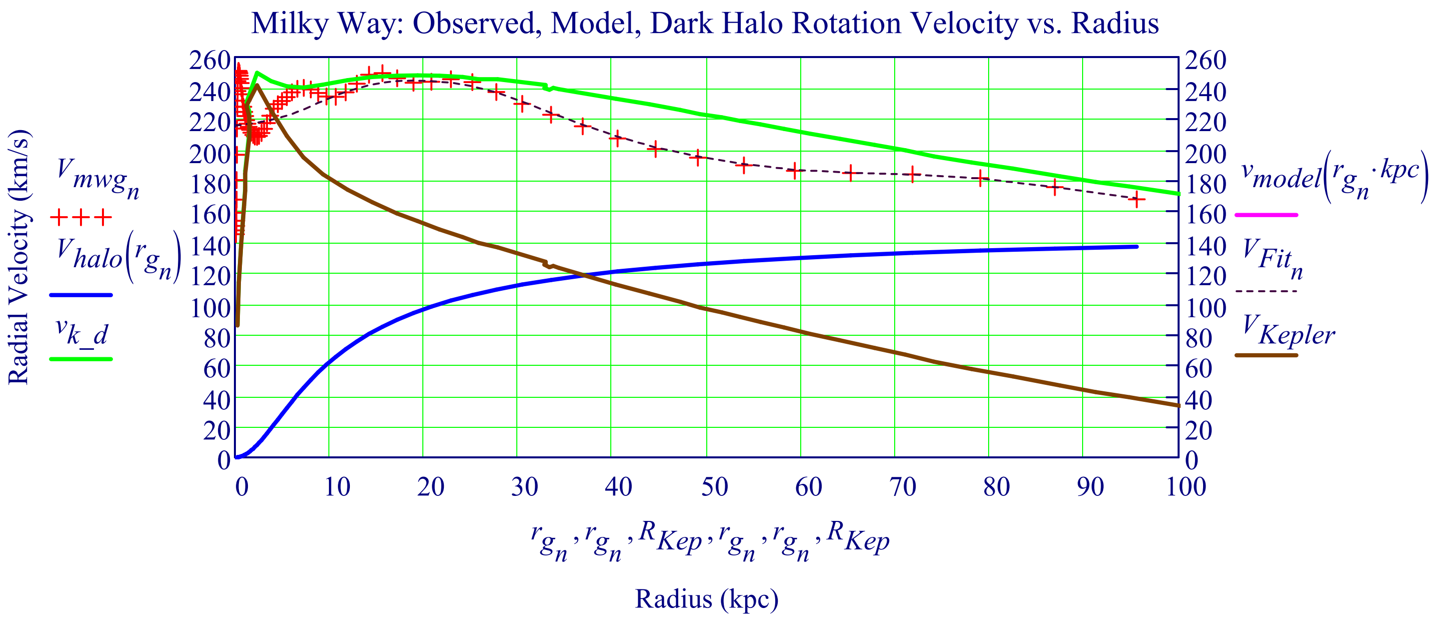
Estimate of Dark Halo - Isothermal Spherical Distribution

$$V_{inf} := 150 \frac{km}{s} \qquad V_{halo}(r) := V_{inf} \cdot \left(1 - \frac{R_h}{r \cdot kpc} \cdot atan\left(\frac{r \cdot kpc}{R_h}\right)\right) \cdot \frac{1}{\frac{km}{s}}$$

Sum of Keplerian and Dark Halo Distributions

$$v_{k\_d} := \overrightarrow{V_{Kepler} + V_{halo}(R_{Kep})}$$

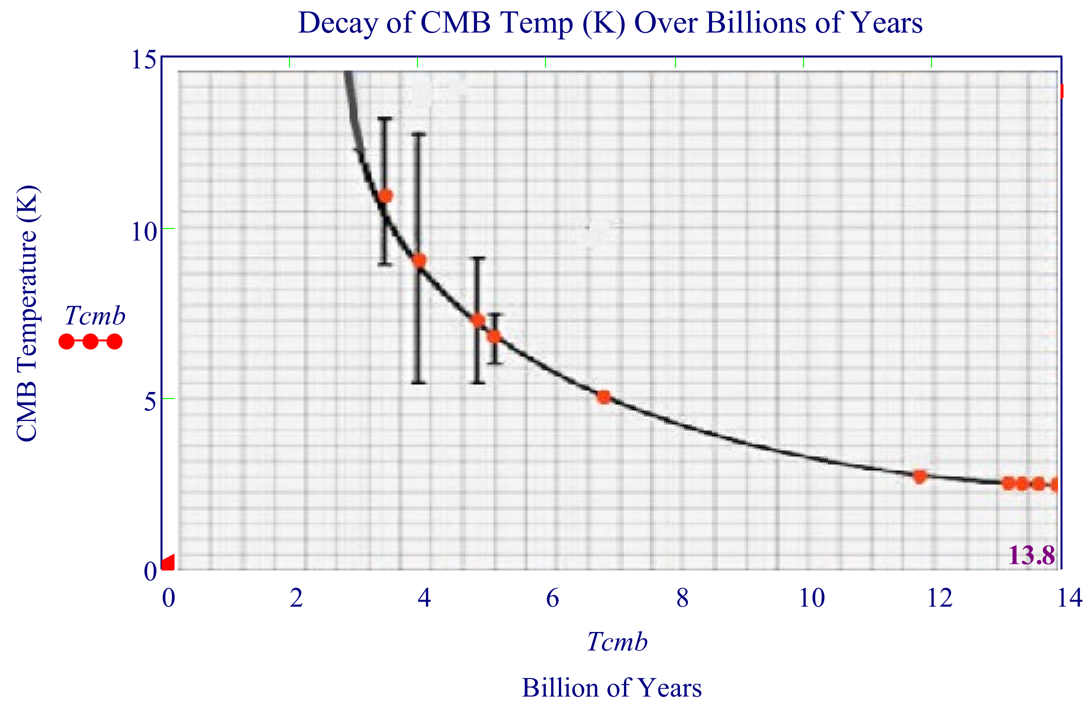
Velocity Plots: Milky War Data (++), Vhalo of Dark Matter (Blue), vk_d Sum of Dark and Kepler, Galaxy Model (Purple), VFit Fit Curve to Data+ (Dashed Black), VKep Kepler Plot (Red)



XXI. Evidence for Λ-CDM "Big Bang" Model

What are the strongest physical evidences for the big bang?

Thanks to technological advances, astronomers can measure the current temperature of radiation lingering from the cosmic origin event as well as the temperatureof this radiation at various times in the past. As the figure below shows, actual temperature measurements match the cooling curve a big bang model (creation model)predicts, given the age of the cosmos (~13.8 billion years old) and its measured expansion rate. The most accurate of these past measurements is the one in the middle of the cooling curve. This measurement fits the curve so closely that its error bar can't be seen in this graph. Figure 2: Evidence of Cooling from the Big Bang Creation Event. The curve is the predicted cooling of the universe according to the big bang creation model with a cosmicage of 13.79 billion years and an average cosmic expansion rate at 68.65 kilometers/second/megaparsec.The dots and error bars are actual temperature measurements of the Cosmic Microwave Background Radiation.



Does the Law of Conservation of Energy Apply to the Big Bang.

As the Universe expands, Dark Energy is created. Energy by itself is not conserved. Energy can increase or decrease whenever space itself changes in time. Photons have an energy that is inversely proportional to their wavelength. As space expands, the wavelength of photons increases and it energy decreases. So where it go? This is why the Cosmic Microwave Background Radiation is so cold. In GR, we have a more complicated theory of Energy Conservation.

Generalized Energy Conservation

It Generalized Energy Conservation of Covariant Conservation Law of the Stress-Energy Tensor. The change in energy in the photon has to match the change in energy of space.

Comparison of Theoretical (Ideal) vs. Measured CMB Temp. from Very Large Telescope, VLT

Data Source: *The evolution of the cosmic microwave background temperature Measurements of T CMB at high redshift from carbon monoxide excitation,* P. Noterdaeme , P. Petitjean , R. Srianand, C. Ledoux, and S. López

A milestone of modern cosmology was the prediction and serendipitous discovery of the cosmic microwave background (CMB), the radiation leftover after decoupling from matter in the early evolutionary stages of the Universe. A prediction of the standard hot Big-Bang model is the linear increase with redshift of the black-body temperature of the CMB (T_{CMB}). This radiation excites the rotational levels of some interstellar molecules, including carbon monoxide (CO), which can serve as cosmic thermometers. Using three new and two previously reported CO absorption-line systems detected in quasar spectra during a systematic survey carried out using Very Large Telescope, VLT / European Southern Observatory, UVES, we constrain the evolution of T_{CMB} to $z \sim 3$. Combining precise measurements with previous constraints, we obtain $T_{CMB}(z) = (2.725 \pm 0.002) \times (1+z)^{1-ß}$ K with $ß = -0.007 \pm 0.027$, a more than two-fold improvement in precision. The measurements are consistent with the standard (i.e. adiabatic, $ß = 0$) Big-Bang model and provide a strong constraint on the effective equation of state of decaying dark energy (i.e. $w_{eff} = -0.996 \pm 0.025$).

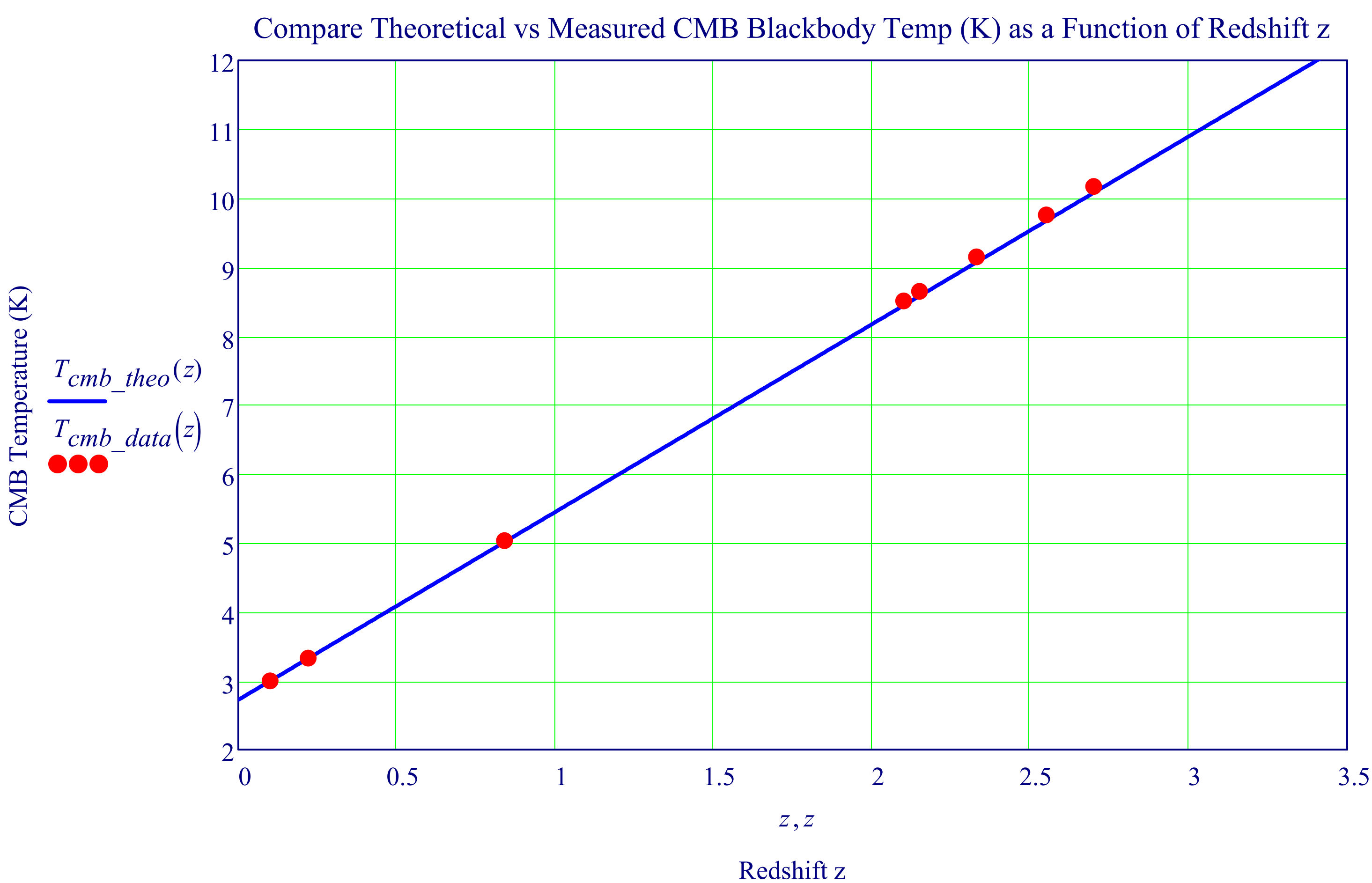
Theoretical (Ideal) CMB Temperature vs Redshift z

$$T_{cmb\_theo}(z) := 2.725 \cdot (1+z)$$

Measured CMB Temperature vs Redshift z

$$\beta := -0.007 \qquad T_{cmb\_data}(z) := 2.725 \cdot (1+z)^{1-\beta}$$

Measurements are based on the rotational excitation of CO molecules are represented by red dots.



XXII. Λ-CDM Model Parameters

http://astro.vaporia.com/start/lambdacdm.html

Wikipedia

"The current standard model of cosmology, the Lambda-CDM model, uses the FLRW metric. By combining the observation data from some experiments such as WMAP and Planck with theoretical results of Ehlers–Geren–Sachs theorem and its generalization, astrophysicists now agree that the early universe is almost homogeneous and isotropic (when averaged over a very large scale) and thus nearly a FLRW spacetime. That being said, attempts to confirm the purely kinematic interpretation of the Cosmic Microwave Background (CMB) dipole through studies of radio galaxies and quasars show disagreement in the magnitude. Taken at face value, these observations are at odds with the Universe being described by the FLRW metric. Moreover, one can argue that there is a maximum value to the Hubble constant within an FLRW cosmology tolerated by current observations, $H_0 = 71 \pm 1$ km/s/Mpc, and depending on how local determinations converge, this may point to a breakdown of the FLRW metric in the late universe, necessitating an explanation beyond the FLRW metric."

parameter	symbol	determined value
physical baryon density	$\Omega_b h^2$	0.02230
physical dark matter density	$\Omega_d h^2$	0.1188
age of the universe	t_0	13.799×10^9 years
scalar spectral index	n_s	0.9667
curvature fluctuation amplitude	Δ^2_R	2.441×10^{-9}
optical depth to the epoch of reionization (EOR)	τ	0.066

item	symbol	calculated value
Hubble constant	H_0	67.74 km $s^{-1}Mpc^{-1}$
baryon density parameter	Ω_b	0.0486
dark matter density parameter	Ω_d	0.2589
matter density parameter	Ω_m	0.3089
dark energy density parameter	Ω_Λ	0.6911

Parameters for Galactic mass components	
Component, Parameter, Value, Uncertainty	
Bulge Mass M b = 1.80 × 10 10 M ⊙ ~ 5 %	**Bar for 3 kpc dip** Amplitude δ bar > 0.8 × Σ d —
Half-mass scale radius R b = 0.5 kpc	Assumed bar half length † 1.7 kpc —
SMD at R b Σ be = 3.2 × 10 3 M ⊙ pc −2	Assumed tilt angle † 13 ◦ —
Center SMD Σ bc = 6.8 × 10 6 M ⊙ pc −2	**Bulge, disk, rings** Total mass M bdr = 8.3 × 10 10 M ⊙ ~ 5 %
Center volume density ρ bc = ∞ —	**Dark halo**
Disk Mass M d = 6.5 × 10 10 ~ 5 %	Mass in r = 10kpc sphere M h (10kpc) = 4.2 × 10 10 M ⊙ ~ 10 %
Scale radius R d = 3.5 kpc	(Sphere) Mass in r = 20 kpc sphere ‡ M h (20kpc) = 1.24 × 10 11 M ⊙
Center SMD Σ dc = 8.44 × 10 2 M ⊙ pc −2	Core radius R h = 5.5 kpc
Center volume density ρ dc = 8M ⊙ pc −3	Central SMD in | z | < 10 kpc Σ hc = 352M ⊙ pc −2
Rings Mass M r ~ 0	Central volume density ρ hc = 0.03M ⊙ pc −3
Peak Σ r 0.17 and 0.34 × Σ d ~ 20 %	Circular velocity at infinity V ∞ = 200km s-1
Radii of wave nodes R r = 3 and 9.5 kpc ~ 3	**Total Galactic mass**
Widths w r = 1 and 2 kpc ~ 10	Mass in r = 20 kpc sphere M total (20kpc) = 2.04 × 10 11 M ⊙ ~ 10 %

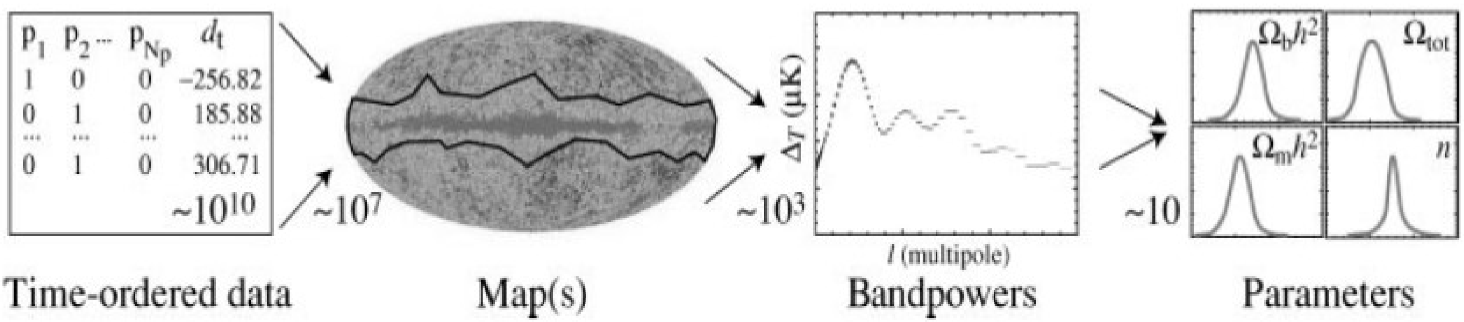
CMB Data Analysis Methodology: Angular Temperature Power Spectrum (TT)

CMB Data Analysis Methodology

Data pipeline and radical compression. Maps are constructed for each frequency channel from the data timestreams, combined, and cleaned of foreground contamination by spatial (represented here by excising the galaxy) and frequency information. Bandpowers are extracted from the maps and cosmological parameters from the bandpowers. Each step involves a substantial reduction in the number of parameters needed to describe the data, from potentially 10'° —> 10 for the Planck satellite.

In every step of CMB data analysis the aim is to reduce the volume of data without losing information.

CMB Data Analysis Pipeline



CMB temperature anisotropies are expressed in terms of multipoles:

If $\delta T/T$ is expanded in terms of **Spherical Harmonics: Y_{lm}**

$$\frac{\delta T(\theta,\phi)}{T_0} = \sum_{l,m} a_{lm} Y_{lm}(\theta,\ \phi)$$

then the complex coefficients a_{lm}, in a homogeneous and isotropic universe, satisfy the condition

$$a_{lm} = \int \frac{\Delta T(n)}{T} \cdot Y_{lm}(n)\, dn$$

$$Y_0^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$Y_1^{-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{-i\phi}$$

$$Y_1^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$$

$$Y_1^1(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{i\phi}$$

Where a_{lm} follow the Gaussian (maximally randomized) distribution with zero mean and variance given by C_l :

$$\langle a^*_{l'm'} a_{lm}\rangle = \delta_{ll'}\,\delta_{mm'}\,C_l$$

It is the **variance of the temperature field** which **carries the cosmological information**, rather than the values of the individual $a_{\ell m}$s; in other words the power spectrum in ℓ fully characterizes the anisotropies. The power at each ℓ is $(2\ell+1)C_\ell/(4\pi)$, and a statistically isotropic sky means that all ms are equivalent.

An unbiased estimator of C_l is defined as:

$$C_l = \frac{1}{2l+1}\cdot\sum_{m=-l}^{l}\left(a_{lm}\cdot a^*_{lm}\right)$$

CMB Likelihood

- CMB temperature and polarization observations can constrain cosmological parameters if the likelihood function can be computed exactly.
- Computing the likelihood function exactly in a brute force way is computationally challenging since it involves inversion of the covariance matrix i.e., $O(N^3)$ computation.
- In Cosmological parameter estimation a theoretical model is represented by its angular power spectrum C_l .
- For a set cosmological parameters we can compute the angular power spectrum C_l using publicly available Boltzmann codes like **CMBFAST** and **CAMB** (Code for Anisotropies in the Microwave Background) and try to fit that with observed C_l . **CMBquick** (Refer to Section XVI) is implemented in Mathematica.

XXIII. Planck Microwave Anisotropy Probe CMB Angular Temp. Power Spectrum (TT)

The Wilkinson Microwave Anisotropy Probe (WMAP) was lauched in 2001.
Planck, launched in 2009, images the sky with more than 2.5 times greater resolution than WMAP.

https://irsa.ipac.caltech.edu/data/Planck/release_3/ancillary-data/cosmoparams/

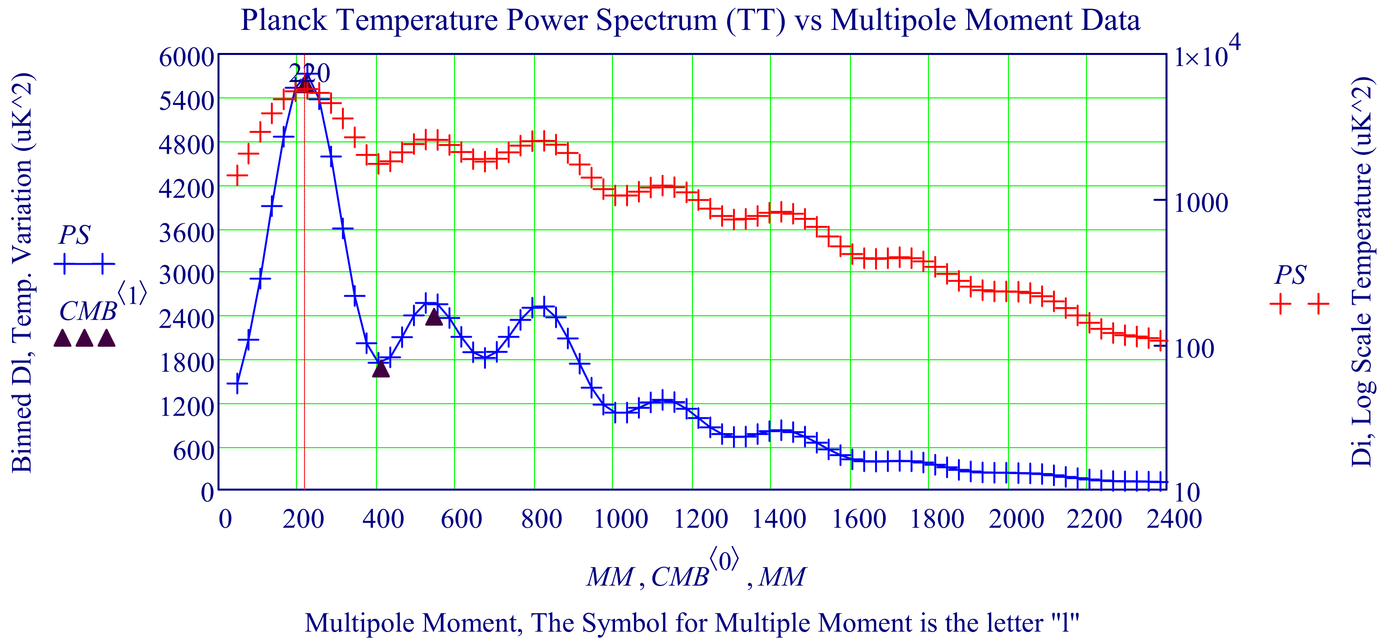
$Planck\_tt := READPRN(\text{"COM\_PowerSpect\_CMB-TT-binned\_R3.01.txt"})$

$MM := Planck\_tt^{\langle 0 \rangle} \quad PS := Planck\_tt^{\langle 4 \rangle}$

CMB Table Has Peaks & Troughs

See CMB Table Below

CMB **Holds a Matrix of Values**



WMAP: TT AND TE ANGULAR POWER SPECTRUM PEAKS FOR ABOVE SPECTRUM

The Characteristics of the Above Spectrum Reveals the Values Needed to Model BB Cosmology

Baryonic fraction $M_{b+d} = M_{b+d+h} = 0.072$

CMB Peaks and Troughs Table

Quantity	Symbol	ℓ	ΔT_ℓ (μK)	ΔT_ℓ^2 (μK^2)	FULL ℓ	FULL ΔT_ℓ^2 (μK^2)
First TT peak	ℓ_1^{TT}	220.1 ± 0.8	74.7 ± 0.5	5583 ± 73	219.8 ± 0.9	5617 ± 72
First TT trough	$\ell_{1.5}^{TT}$	411.7 ± 3.5	41.0 ± 0.5	1679 ± 43	410.0 ± 1.6	1647 ± 33
Second TT peak	ℓ_2^{TT}	546 ± 10	48.8 ± 0.9	2381 ± 83	535 ± 2	2523 ± 49
First TE antipeak	ℓ_1^{TE}	137 ± 9	...	-35 ± 9	151.2 ± 1.4	-45 ± 2
Second TE peak	ℓ_2^{TE}	329 ± 19	...	105 ± 18	308.5 ± 1.3	117 ± 2

Based on the the spatial variation of the CMB and the Model Parameters of the Λ-CDM , astrophysicists predicted a Hubble Constant of 67.5 ±0.5 km/s per megaparsec. This is different from the Hubble Constant value measured from the change of recessional velocity of galaxies with distance.

XXIV. James Webb Space Telescope (JWST)

The James Webb Space Telescope (JWST) is the scientific successor to both the Hubble Space Telescope and the Spitzer Space Telescope. It is envisioned as a facility-class mission. JWST aims to achieve science goals that can never be reached from even the largest envisioned groundbased telescopes.
It will be equipped with four instruments capable of studying the **0.6 to 28μm region** using both imaging and spectroscopic techniques. The instrument suite provides broad wavelength coverage and capabilities aimed at four key science themes:

1) The End of the Dark Ages: First Lig.ht and Reionization; finding the light from the first objects to coalesce after the Universe has cooled after the Big Bang
2) The Assembly of Galaxies; how do galaxies change from first light objects to the suite of morphologies and galaxy types that we see today. To unravel the birth and early evolution of star, from the earliest epochs ≲ 300 Myr after the Big Bang, through the Epoch of Reionization.
3) The Birth of Stars and Protoplanetary Systems;
4) Planetary Systems and the Origins of Life. NIRCam is the 0.6 to 5 micron imager for JWST, and it is also the facility wavefront sensor used to keep the primary mirror in alignment. JWST will work to unravel the birth and early evolution of stars, from infall onto dust-enshrouded protostars to the genesis of planetary systems.

JWST Instruments

JWST Mid Infrared Instrument

The JWST Mid-Infrared Instrument (**MIRI**) providesimaging and spectroscopic observingmodes from~5 to 28μm.

JWST Near Infrared Camera

The JWST Near Infrared Camera (**NIRCam)** offers imaging, coronagraphy, wide field slitless spectroscopy, and time-series monitoring both in imaging and spectroscopy, as well as wavefront sensing measurements for JWST mirror alignment.
The JWSTprovides near-IR spectroscopy from 0.65.3 μmwithin a 3.4 ×3.6 arcmin field of view using a micro-shutter assembly (MSA), an integral field unit (IFU), and fixed slits (FSs).

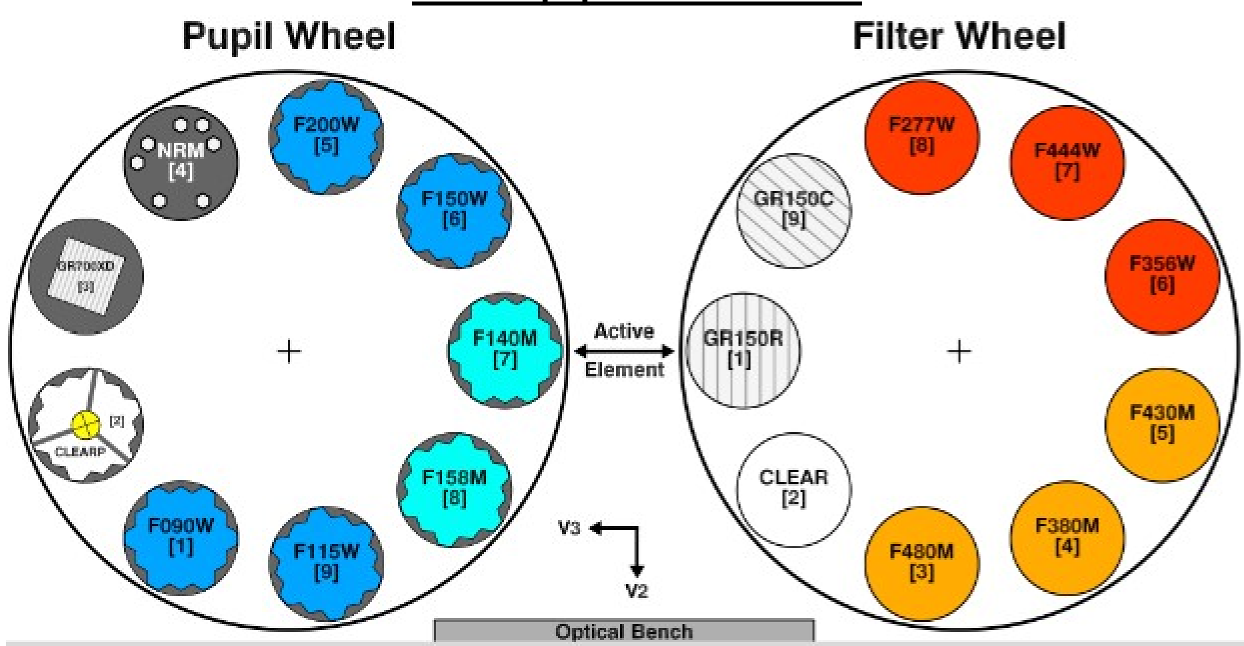
JWST Near Infrared Imager and Slitless Spectrograph

The JWST Near Infrared Imager and Slitless Spectrograph (NIRISS) provides observing modes for slitless spectroscopy, high-contrast interferometric imaging, and imaging, at wavelengths between 0.6 and 5.0 μm over a 2.2' x 2.2' FOV.

JWST Near Infrared Spectrograph

The JWST Near Infrared Spectrograph (NIRSpec) provides near-IR spectroscopy from 0.6–5.3 μm within a 3.4 × 3.6 arcmin field of view using a micro-shutter assembly (MSA), an integral field unit (IFU), and fixed slits (FSs).

NIRISS pupil and filter wheels



JADES: JWST Advanced Deep Extragalactic Survey Near-IR Spectroscopy Optics

JADES: Lookback Time versus Red Shift and Age of Univ z = 13.2 Gyr

Look-Back Time & Age of Unuv vs. z. 2023 Metal-Poor JADES-GS-z13-0 galaxy @z=13.2, Age:13.4 Gyr

The Value of the Cosmological Constant, John D. Barrow, 2018

If you neglect the energy density of radiation and consider that the universe is currently flat, the following formula is derived from the Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{K}{a^2},$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}$$

$$\dot{\rho} + 3H(P + \rho) = 0,$$

$H_0 := 71 \frac{km}{s \cdot Mpc}$

$\Omega_{\Lambda 0} := 0.73$

$$dt = \frac{da}{H_0\left(\frac{\Omega_{m,0}}{a} + a^2\Omega_{\Lambda,0}\right)^{\frac{1}{2}}}$$

The subindices mean current values for the Hubble parameter (= 71 Km /s Mpc), Omega matter (= 0.27), Omega cosmological constant (= 0.73). To get the age at a given redshift z, you have to integrate from a = 0, to a = 1/(1+z).

The fraction of the effective mass of the universe attributed to "dark energy" or the cosmological constant is $\Omega_{\Lambda 0}$ With 73% of the influence on the expansion of the universe in this era, the dark energy is viewed as the dominant influence on that expansion.

We assumethat the matter source of the FLRW universe is a perfect fluid with energy density ρ and pressure P related by the barotropic, linear, and constant equation of state **P = wρ, w = const.**

$$z = \frac{\lambda_{observed} - \lambda_{expected}}{\lambda_{expected}}$$

$j := 0, 0.01 .. 20$

$Lyr := 1yr \cdot c$ $\quad Lyr = 9.467 \times 10^{15} m$ $\quad Mpc := 3.086 \cdot 10^6 Lyr$ $\quad Gyr := 10^9 yr$ $\quad w := 0.1 .. 1$ $\quad z_j := j$

Note: t_L(z) factor should be 3/2. Used 1.45 to get a better match.

$$t_L(z) := \frac{3}{2H_0 \cdot 1.45} \cdot \left[1 - (1 + z)^{-\frac{3}{2}}\right]$$

w: Ratio P/ρ for a fluid

$$t_0(w) := \frac{2 \cdot H_0^{-1}}{3(w+1) \cdot \sqrt{\Omega_{\Lambda 0}}} \cdot ln\left(\frac{1 + \Omega_{\Lambda 0}}{\sqrt{1 - \Omega_{\Lambda 0}}}\right)$$

Furthest Observations of 2023 Metal-Poor JADES-GS-z13-0 galaxy @z=13.2, 13.4 Gyr Ago

Data: https://en.wikipedia.org/wiki/Age_of_the_universe#/media/File:Look-back_time_by_redshift.png

$LBT\_z :=$ READPRN("Look-back Time by Redshift z =13.2 Galaxies.csv")

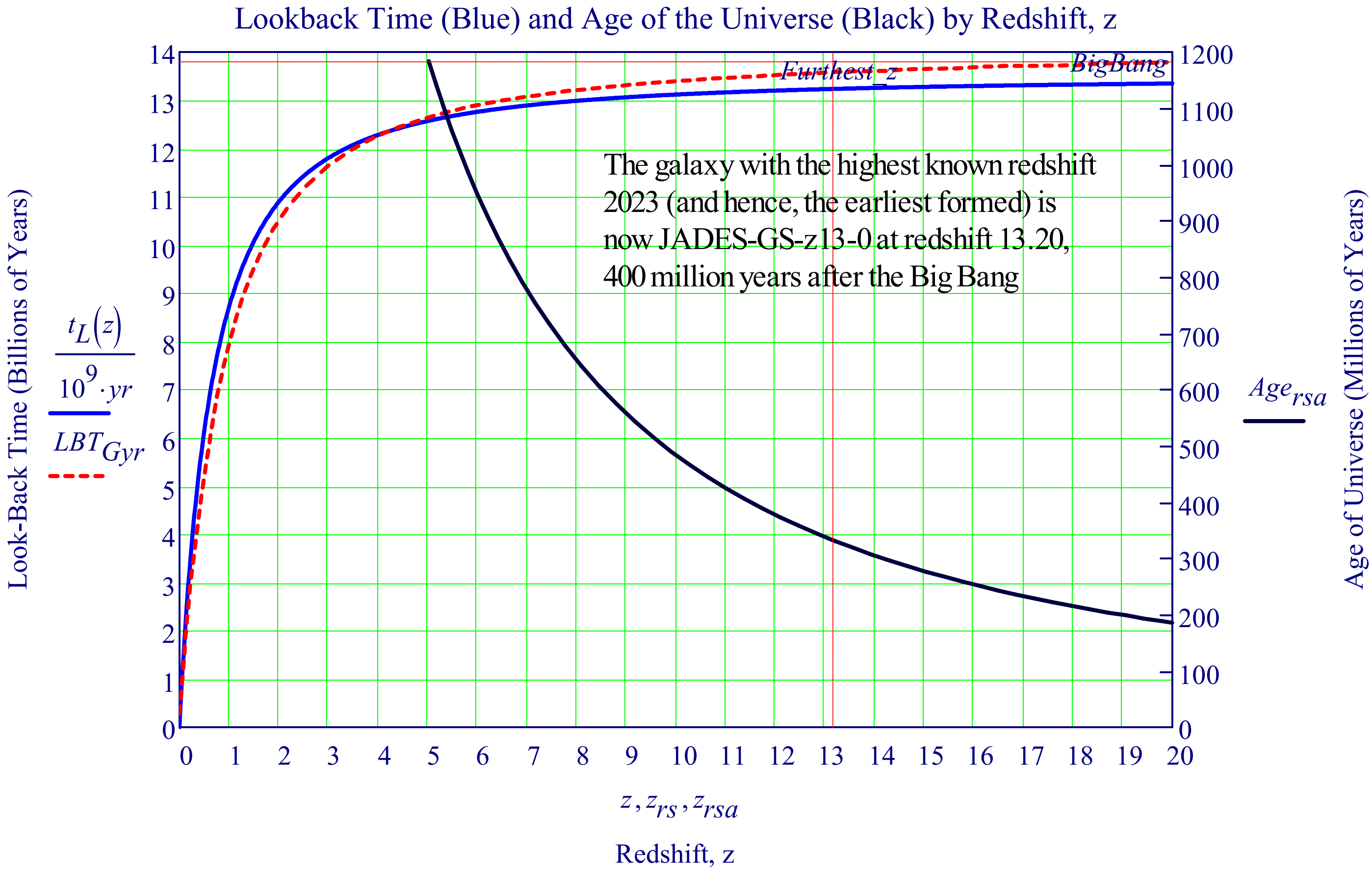
$Age\_RS :=$ READPRN("Age of Universe by Redshift.csv")

$z_{rs} := LBT\_z^{\langle 0 \rangle}$ $\quad LBT_{Gyr} := LBT\_z^{\langle 1 \rangle}$ $\quad z_{rsa} := Age\_RS^{\langle 0 \rangle}$ $\quad Age_{rsa} := Age\_RS^{\langle 1 \rangle}$

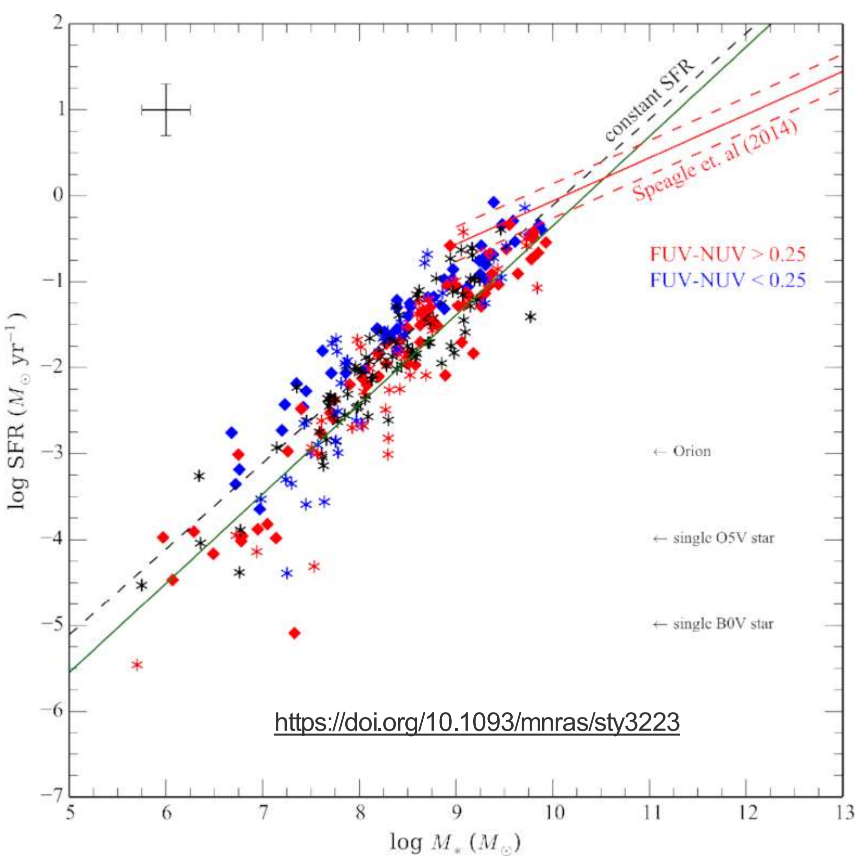
JWST Advanced Deep Extragalactic Survey - Lookback Time

$Furthest\_z := 13.2$ $\quad BigBang := 13.8$

Look-back Time by Reshift and Age of Universe



The mass-to-light ratios and the star formation histories of Disc Galaxies



The main sequence for high- and low-mass star-forming galaxies.

Star Formation Rate, SFR, is the rate at which gas and dust is turned into stars. It is the total mass of stars formed per year. The term can be used in describing a galaxy or globular cluster. **Data: 2017**

The data sets from Cook et al. (the solid symbols) and LSB + SPARC (the starred symbols) are shown, colour coded by FUV–NUV color.

The green line is a fit to the LSB + SPARC sample (McGaugh, Schombert & Lelli 2017).

The dashed line is the line of constant star formation for a 13 Gyr Universe.

There is a clear trend for **blue FUV −NUV colours** to lie **above** the constant SFR line (rising SFR in the last 100 Myr) versus **red FUV −NUV colours below the line** (declining SFH).

The z = 0 relationship from Speagle et al. is shown for the high-mass spirals, along with 3σboundaries.

Also shown are the values for SFR that correspond to an Orion-sized complex, a single O star and a single B star.

SFR estimates below −4.5 are highly inaccurate.

A representative error is shown in the upper left, errors in SFR and stellar mass are from McGaugh, Schombert(2017).

Latest Findings JWST Challenge Cosmology Models

Astrophysicists may have an explanation for the James Webb Space Telescope's discovery of a swarm of mysterious early galaxies that threaten to break cosmology.

The Big Bang Model predicts that, as we look farther and farther back in time — i.e., to greater and greater cosmic distances — that the galaxies we see will be inherently smaller, bluer, less evolved, less rich in heavy elements, and that at some point beyond where we've been able to look, we should cease to see stars or galaxies of any type, as we'll reach the Universe's "dark ages."

https://www.livescience.com/space/cosmology/james-webb-telescopes-observations-of-impossible-galaxies-at-the-dawn-of-time-may-finally-have-an-explanation
The galaxies, which the James Webb telescope (JWST) spotted forming as early as 500 million years after the Big Bang, were so bright that they theoretically shouldn't exist: Brightnesses of their magnitude should only come from massive galaxies with as many stars as the Milky Way, yet these early galaxies took shape in a fraction of the time that ours did.

The discovery threatened to upend physicists' understanding of galaxy formation and even the standard model of cosmology. Now, a team of researchers using supercomputer simulations suggest that the galaxies may not be so massive at all — they could just be unusually bright.

Bursts of star formation explain mysterious brightness at cosmic dawn Intense ashes of light, not mass, resolve the puzzle of impossible brightness Peer-Reviewed Publication,NORTHWESTERN UNIVERSITY, 3-OCT-2023
A period that lasted from roughly 100 million years to 1 billion years after the Big Bang, cosmic dawn is marked by the formation of the universe's rst stars and galaxies. Before the JWST launched into space, astronomers knew very little about this ancient time period.

"The JWST brought us a lot of knowledge about cosmic dawn," Sun said. "Prior to JWST, most of our knowledge about the early universe was speculation based on data from very few sources. With the huge increase in observing power, we can see physical details about the galaxies and use that solid observational evidence to study the physics to
understand what's happening."

Do JWST's results contradict the Big Bang?
https://bigthink.com/starts-with-a-bang/jwsts-contradict-big-bang/

Many of these early galaxies that JWST is finding have peculiar, puzzling properties about them that appear difficult to reconcile with this theoretical picture that the Universe has painted for us. They appear, for example, to be:

- very massive,
- very bright,
- very rich in heavy elements,
- very actively forming new stars,
- and very rich in gas.

Prognosis:
There are an enormous number of astrophysical possibilities that invoke no fundamentally new physics that could potentially account for why these galaxies would exist with these large masses and brightnesses.

XXV. Mathematica CMBquick: Simulation of CMB Temperature Power Spectrum

WMAP Temperature Power Spectrum (TT) vs Multipole Moment Modeling

This Analysis is Based on Cyril Pitrou's Mathematica tools for creating CMB Spectra.

https://www2.iap.fr/users/pitrou/

"CMBquick is a package for Mathematica in which tools are provided to compute the spectrum and bispectrum of Cosmic Microwave Background (CMB)... CMBquick is a slow but precise and pedagogical, tool which can be used to explore and modify the physical content of the linear and non-linear dynamics. Second, its is a tool which can help developing templates for nonlinear computations, which could then be hard coded once their correctness is checked. The number of equations for non-linear dynamics is quite sizable and CMBquick makes it easy (but slow) to manipulate the non-linear equations, to solve them precisely, and to plot them."

Below are the results of CMBquick Simulation to find the Temp Power Spectrum for WMAP

Compare The Analysis Results Below to the Analysis from the Previous Section, XV

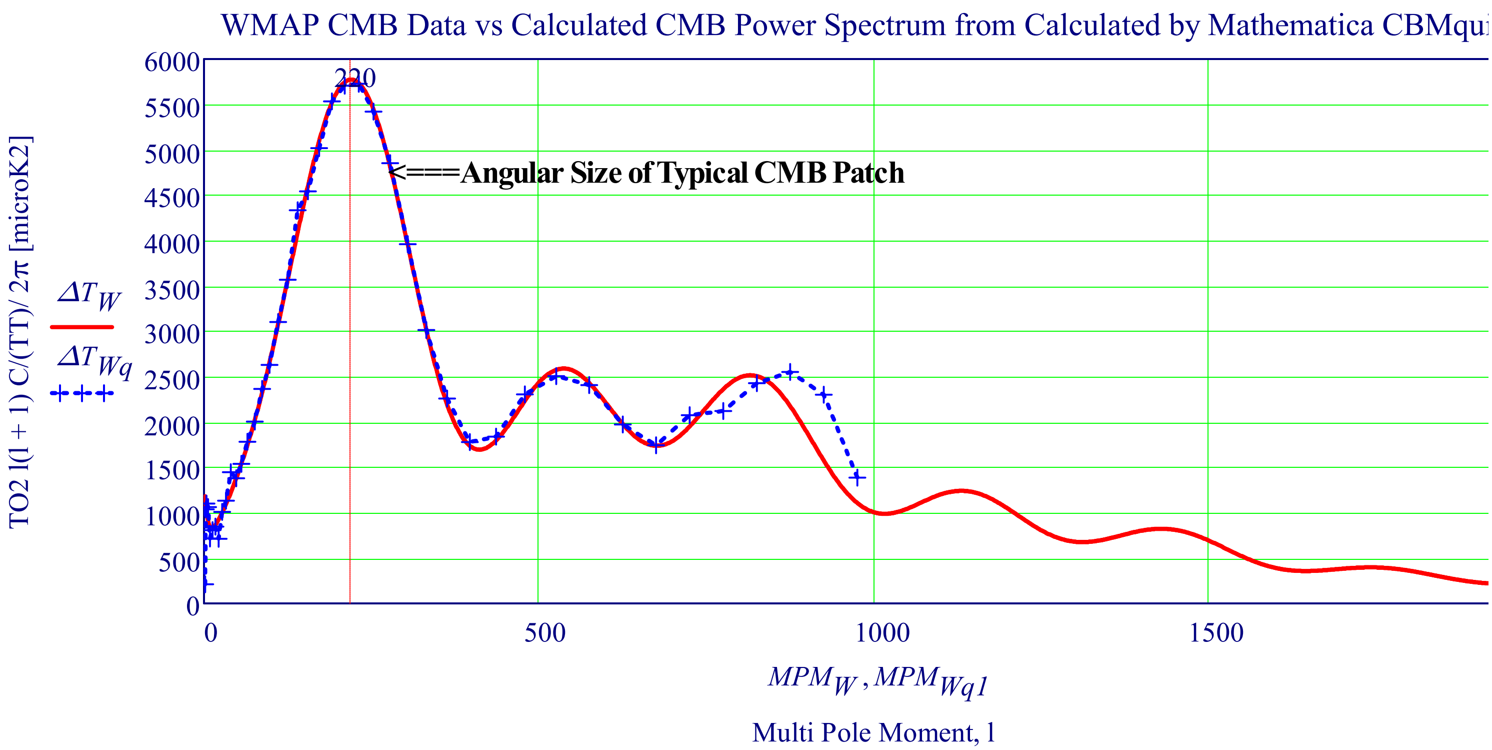
WMAP Temperature Power Anistrophies Calculated from Mathematica (CMB quick)

$WMAP\_CMB := READPRN(\text{"CAMB\_WMAP-CMBquick.dat"})$

$WMAP\_CMBq := READPRN(\text{"wmap\_CMBq.dat"})$

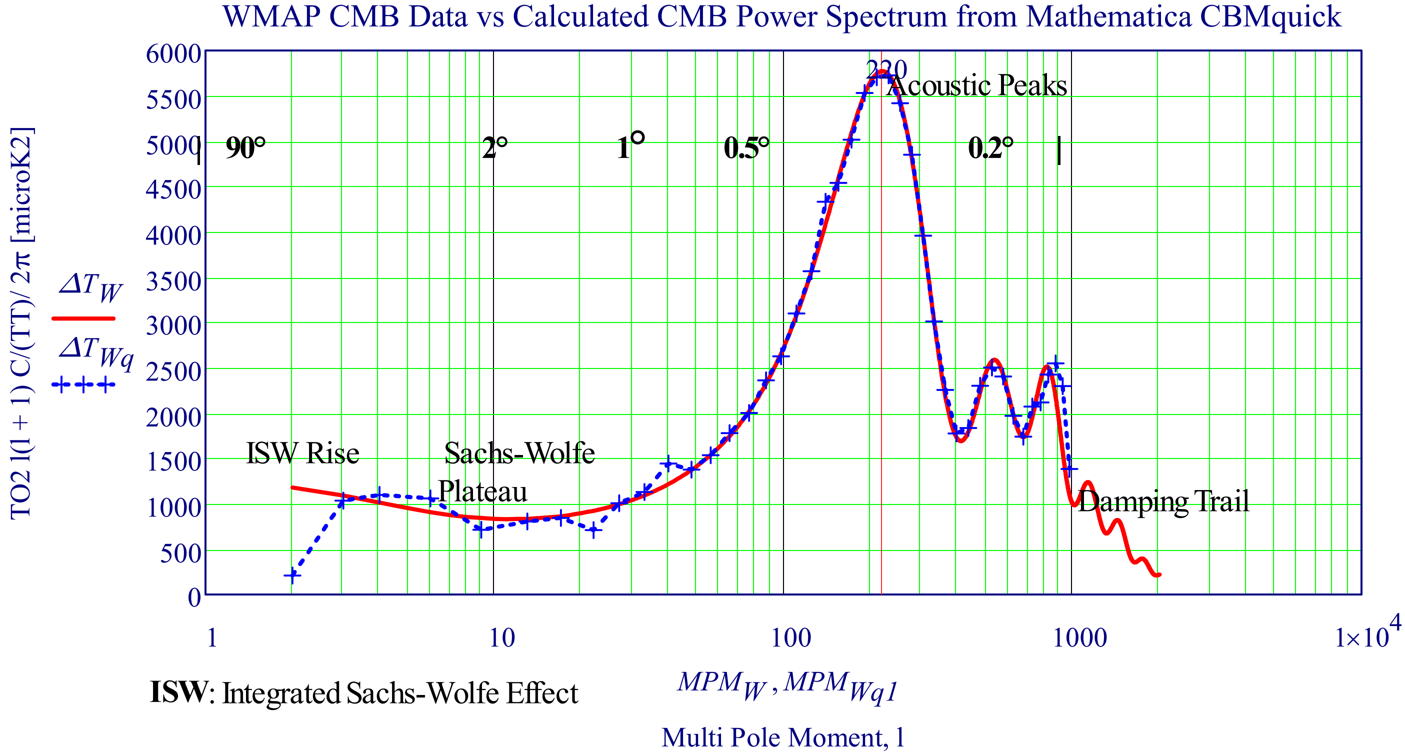
$\Delta T_W := WMAP\_CMB^{\langle 1 \rangle}$ $\quad$ $MPM_W := WMAP\_CMB^{\langle 0 \rangle}$

$\Delta T_{Wq} := WMAP\_CMBq^{\langle 3 \rangle}$ $\quad$ $MPM_{Wq1} := WMAP\_CMBq^{\langle 0 \rangle}$



Angular Scale ° and Projection Effects on CMB

The corresponding angle on the sky is approximately 100 / *l* degrees°



ISW: Integrated Sachs-Wolfe Effect

Projection Effects

See Pages 9, 10 and 11 for Definitions

$$\rho_0 = \frac{3 \cdot H_0^{\,2}}{8 \cdot \pi \cdot G} \qquad \rho_0 := 8.6443584621592 \cdot 10^{-27} \cdot \frac{kg}{m^3} \qquad \rho_{crit} = 1.879\ \ 10^{-29}\,h^2\,g\,cm^{-3}.$$

$$\Omega_M = \frac{8\pi G \cdot \rho}{3 \cdot H_0^{\,2}} \qquad \Omega_\Lambda = \frac{\Lambda \cdot c^2}{3 \cdot H_0^{\,2}}$$

$\Omega_0 = 1$ Radiation Transfer Function

The mass is usually parameterized by Ω_0 which is the energy density in units of the critical density

$$\Omega_0 = \frac{\rho_T}{\rho_{crit}} \qquad \Omega_\Lambda = \frac{\rho_\nu}{\rho_{crit}} \qquad \Omega_0 + \Omega_\Lambda = 1$$

ρ_v is the vacuum contribution

$\Omega_0 > 0.1$ to 0.3 $\qquad \Omega_0 := 0.15$ $\qquad$ The Baryon Fraction is $\Omega_b h^2 = 0.01$ to 0.02

For the acoustic contributions, the k modes that reach extrema in their oscillation at last scattering form a harmonic series of peaks related to the sound horizon. This in turn is approximately

$$\frac{\eta_{star}}{\sqrt{1 + C \cdot \left(1 + R\left(\eta_{star}\right)\right)}} \qquad R\left(\eta_{star}\right) = 30\Omega_b \cdot h^2 \qquad C := \sqrt{3} - 1$$

Since $\Omega_b h^2$ must be low to satisfy nucleosynthesis constraints, the sound horizon will scale roughly as the particle horizon . The particle horizon at last scattering itself scales as

$$\eta_{star} = \left(\Omega_0 \cdot h^2\right)^{\frac{1}{2}} \cdot f_R \qquad f_R = \sqrt{\left[1 + \left(24\,\Omega_0 \cdot h^2\right)^{-1}\right]} - \sqrt{24\,\Omega_0 \cdot h^2}$$

CMBquick Cosmology CPLP Planck Perturbation Parameters

We compute the cosmology k dependent Boltzman Hierarchy

Variable	Value	Units	
Ω_{b0}	0.049169		Abundance of baryons
Ω_{c0}	0.26474		Abundance of CDM
Ω_{r0}	0.000092414		Abundance of radiation (massless ? 's and photons)
$\Omega_{\Lambda 0}$	0.686		Abundance of Λ
Ω_K	0		Abundance of curvature
T_0	2.7255	K	Temperature of CMB
N_ν	3.045		Number of massless neutrinos
h	0.6727		Reduced H constant
τ_{rei}	0.079		Optical depth of reionization
n_s	0.9645		Scalar perturbations spectral index
k_{eq}	0.010362	Mpc^{-1}	k at equivalence time
z_{rei}	10.701		Redshift at reionization
z_{eq}	3395.7		Redshift at equivalence
z_{LSS}	1061.2		Redshift at t-trei = ln(2)
z_{dec}	1090.3		Redshift at max of visibility function
z_*	1091.2		Redshift at t-trei = 1
$d_A(z_*)$	13910.	Mpc	Angular distance at z *
$d_A(z_{eq})$	14078.	Mpc	Angular distance at equivalence
D_H	4456.56	Mpc	Hubble distance today
t_0	13.8308	Gyears	Age of the Universe
t_*	371 312.	years	Age of universe at z *
$r_{hor}(z_{dec})$	280.58	Mpc	Radius of horizon at z dec
η_0	14191.	Mpc	Conformal time today
A_s^2	2.4736×10^{-9}		Primordial scalar perturbations amp at k = 0.002 Mpc
n_S	0.9645		Scalar spectral index
r	0		Tensor to Scalar ratio at k = 0.002 Mpc
n_T	1		Tensor spectral index
σ_8	0.84516		Relies on extrapolation of the matter power spectrum

XXVI. Calculation of CMB Power Spectra from Model Parameters - CAMB Tool

Code for Anisotropies in the Microwave Background [CAMB].
An Online CAMB Calculation Routine to calculate CMB_Model ΛCDM Model Parameters is available at:
https://lambda.gsfc.nasa.gov/toolbox/camb_online.html

Cosmological Model Parameters for Model Input

Ω_b h^2 = 0.022600
Ω_c h^2 = 0.112000
Ω_ν h^2 = 0.000640
Ω_Lambda = 0.724000
Ω_K = 0.000000
Ω_m (1-Ω_K-Ω_L) = 0.276000
100 θ (CosmoMC) = 1.039532
N_eff (total) = 3.046000
1 η, g= 1.0153 m_nu*c²/k_B/T_nu0= 353.71 (m_nu= 0.060 eV)
Age of universe/GYr = 13.777
z* = 1088.75
r_s(z*)/Mpc = 146.38
100*θ = 1.039819
z_{drag} = 1059.70
r_s(z_{drag})/Mpc = 149.01
k_D(z*) Mpc = 0.1393
100*θ_D = 0.160248
z_EQ (if v_nu=1) = 3216.47
100*θ_EQ = 0.847737
τ_recomb/Mpc = 284.72 τ_now/Mpc = 14362.3

Fake Model Params for Comparison

Ω_b h^2 = 0.05
Ω_c h^2 = 0.112000
Ω_ν h^2 = 0.000640
Ω_Lambda = 0.724000
Ω_K = 0.000000
Ω_m (1-Ω_K-Ω_L) = 0.276000
100 θ (CosmoMC) = 1.039532
N_eff (total) = 3.046000

Fake Model CMB Curve

$CMB\_Model_{Fake} := READPRN($"Lensedcls-CMB Spectrum Om_b h2 050.tx

$$\Delta T2K_{Fake} := CMB\_Model_{Fake}^{\langle 1 \rangle}$$

$$MPM_{Fake} := CMB\_Model_{Fake}^{\langle 0 \rangle}$$

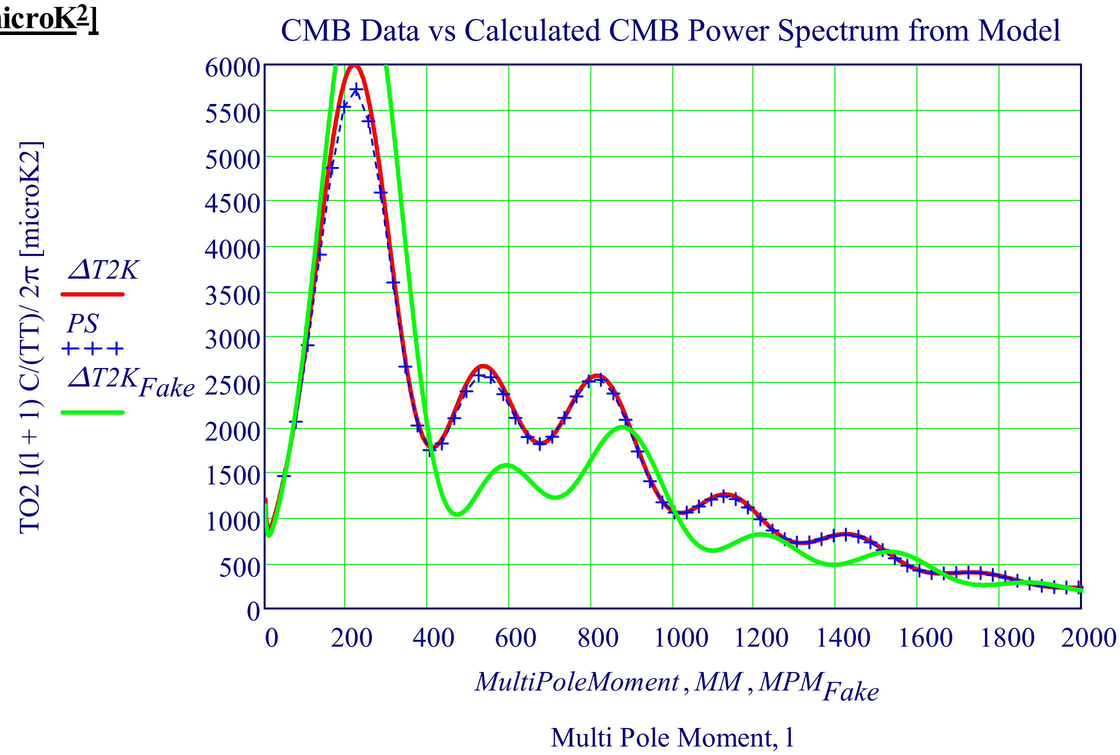
Use the Online CAMB Calculation Routine with Above CMB Parameters ==> ΛCDM Model

$CMB\_Model := READPRN($"Lensedcls-CMB Spectrum.txt"$)$ $\quad rows(CMB\_Model) = 2.099 \times 10^3$

$$\Delta T2K := CMB\_Model^{\langle 1 \rangle} \qquad MultiPoleMoment := CMB\_Model^{\langle 0 \rangle}$$

Note: The Excellent Match Between Data and the Model

TO² $l(l+1)$ C/(TT)/ 2π [microK²]



Calculation of CMB Power Spectra from Model Parameters

This Analysis was taken from *Physical Foundations of Cosmology*, V. Mukhanov, 2005

Chapter 9: Cosmic microwave background anisotropies

After recombination, the primordial radiation freely streams through the universe without any further scattering. An observer today detects the photons that last interacted with matter at redshift z ~1000, far beyond the stars and galaxies. The pattern of the angular temperature fluctuations gives us a direct snapshot of the distribution of radiation and energy at the moment of recombination, which is representative of what the universe looked like when it was a thousand times smaller and a hundred thousand times younger than today. The first striking feature is that the variations in intensity across the sky are tiny, less than 0.01% on average. We can conclude from this that the universe was extremely homogeneous at that time, in contrast to the lumpy, highly inhomogeneous distribution of matter seen today. The second striking feature is that the average amplitude of the inhomogeneities is just what is required in a universe composed of Cold Dark Matter and ordinary matter to explain the formation of galaxies and large-scale structure. Moreover, the temperature autocorrelation function indicates that the inhomogeneities have statistical properties in perfect accordance with what is predicted by hypothetical inflationary models.

The purpose of this chapter is to derive the spectrum of microwave background fluctuations, assuming a nearly scale-invariant spectrum of primordial inhomogeneities, as occurs in inflationary models. Correlation function and multipoles A sky map of the cosmic microwave background temperature fluctuations can be fully characterized in terms of an infinite sequence of correlation functions. If the spectrum of fluctuations is Gaussian, as predicted by inflation and as current data suggest, then only the even order correlation functions are nonzero and all of them can be directly expressed through the two-point correlation function (also known as the temperature autocorrelation function):

$$C(\theta) \equiv \left\langle \frac{\delta T}{T_0}(\mathbf{l}_1)\,\frac{\delta T}{T_0}(\mathbf{l}_2)\right\rangle$$

The temperature autocorrelation function is a detailed fingerprint that can be used first to discriminate among cosmological models and then, once the model is fixed, to determine the values of its fundamental parameters.

Multipole Moments

Spectra tilt, n_s

9.7.4 Calculating the spectrum. We will now proceed to calculate the multipole spectrum $l(l+1)\,C\,l$, $P_\xi(k)$.

The ratio of the value of $l(l+1)\,C\,l$ for $l > 200$ to its value for low multipole moments (the flat plateau) is

$$\frac{l(l+1)\,C_l}{(l(l+1)\,C_l)_{\mathrm{low}\,l}} = \frac{100}{9}(O + N_1 + N_2 + N_3)\,, \qquad (9.109)$$

The contribution to the integrals O in (9.75) and (9.76) arises in the vicinity of the singular point x = 1. N1 is the nonoscillating contribution, N2 and N3 are Doppler contribution to the nonoscillating part of the spectrum. The result in the case of the concordance model ($\mathbf{\Omega_m = 0.3, \Omega_\Lambda = 0.7, \Omega_b = 0.04, \Omega_{tot} = 1}$ and $\mathbf{H = 70\ km\ s^{-1}\ Mpc^{-1}}$) is presented in the Figure below.

The location scale factor, a(ζ,x,t), determines the volume rate of increase:

Fundamental cosmological parameters. To calculate the known history of the homogeneous Universe one needs (in addition to the fundamental constants and the relevant Standard Model parameters) five cosmological parameters. These can be chosen to be the ones above, defined at the present epoch. To describe the inhomogeneity one needs a and n_s, which are shown above specify the spectrum $P_\xi(k)$ and value A of the primordial curvature perturbation ζ. The values of the parameters shown on the above page are chosen so that the calculated CMB spectrum C_ℓ agrees with measurements made in the Planck spacecraft, and are taken from the Planck 2015 results.

$$a(\zeta, x, t) = a(t)\,e^{\zeta(x,t)}$$

$$k_0 := 0.05Mpc^{-1}$$

$$n_s := 1 - 0.35$$

$$P_\xi(k) = A\left(\frac{k}{k_0}\right)^{(n_s-1)}$$

$$P_\xi(k_0) := 2.21\cdot 10^{-9}$$

XXVIIA. The Discovery of the Accelerating Universe (2011)

Distance Modulus vs. Redshift for Type Ia Supernovae from the Supernova Cosmology Project

Lawrence Berkeley National Laboratory Data:
The Supernova Cosmology Project, SCP

Data from: https://supernova.lbl.gov/Union/figures/SCPUnion2.1_mu_vs_z.txt

$SCP := READPRN(\text{"SCPUnion mu vs z. Data Only.txt"})$ $SCP := csort(SCP, 0)$

$z_{mu} := SCP^{\langle 0 \rangle}$ $M_{mu} := SCP^{\langle 1 \rangle}$ $vg := (1\ \ 1\ \ 1)^T$ $rows(SCP) = 580$

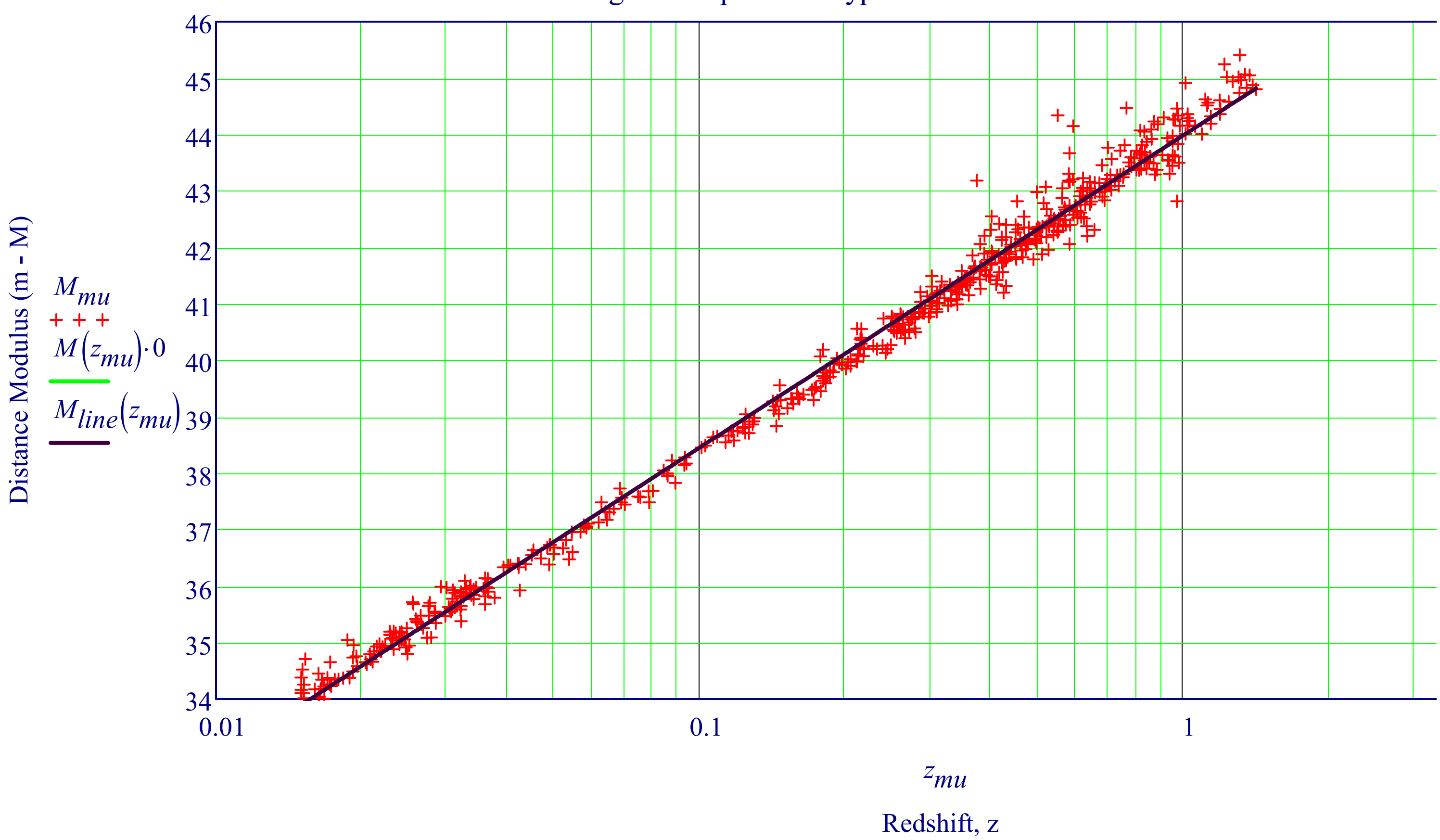
Fit a Logfit Function and a Straight Line to Magnitude vs. Redshift Data

$ab := logfit(z_{mu}, M_{mu}, vg)$ $M(z) := ab_0 \cdot ln(z + ab_1) + ab_2$

$ba := line(log(z_{mu}), M_{mu})$ $M_{line}(z) := ba_0 + ba_1 \cdot log(z)$

$Diff(z) := M(z) - M_{line}(z)$

Hubble Diagram: Supernova Type 1a Measurement - Distance Modulus vs. z



Find the Percent of z > 0.1 Supernovae that are above the Regression Line, Mline

$$PercentAbove := \left(\sum_{n=478}^{579} if\left(M_{mu_n} - M_{line}(z_{mu_n}) > 0, 1, 0 \right) \right) \frac{1}{100}$$

$PercentAbove = 64 \cdot \%$ This 64% shows that the Velocities of the High z Galaxies are statistically increasing faster than the Hubble Constant. **The Expansion is Accelerating.**

XXVIIB. The Discovery of the Accelerating Universe (1999)

Ω AND Λ FROM 42 HIGH-REDSHIFT SUPERNOVAE, Perlmutter et. al. (1999)

Named by Science magazine as the 'Scientific Breakthrough of the Year" for 1998.

The Supernova Cosmology Project, SCP

Attempts to measure the deceleration parameter Λ were stymied for **lack of high-redshift supernovae**. The Supernova Cosmology Project was started in 1988 to address this problem. The primary goal of the project is the determination of the cosmological parameters of the universe using the magnitude-redshift relation of type Ia supernovae. The Project developed techniques, including instrumentation, analysis, and observing strategies, that make it possible to systematically study high-redshift supernovae. As of 1998 March, more than 75 type Ia supernovae at redshifts **z = 0.18 to 0.86** have been discovered and studied by the Supernova Cosmology Project. (Perlmutter et al.)

$z \quad \sigma_z \quad m_X^{peak} \quad \sigma_X^{peak} \quad A_X \quad K_{BX} \quad m_B^{peak} \quad m_B^{eff} \quad \sigma_{m_B^{eff}}$

$ZD := READPRN(\text{"SCP SNE IA DATA - Perlmutter Data Only.txt"}) \qquad rows(ZD) = 42$

$zd := READPRN(\text{"CALAN-TOLOLO SNE IA DATA.txt"}) \qquad rows(zd) = 18$

Merge Data Files: $ZD := stack(zd, ZD) \qquad ZD := csort(ZD, 0)$

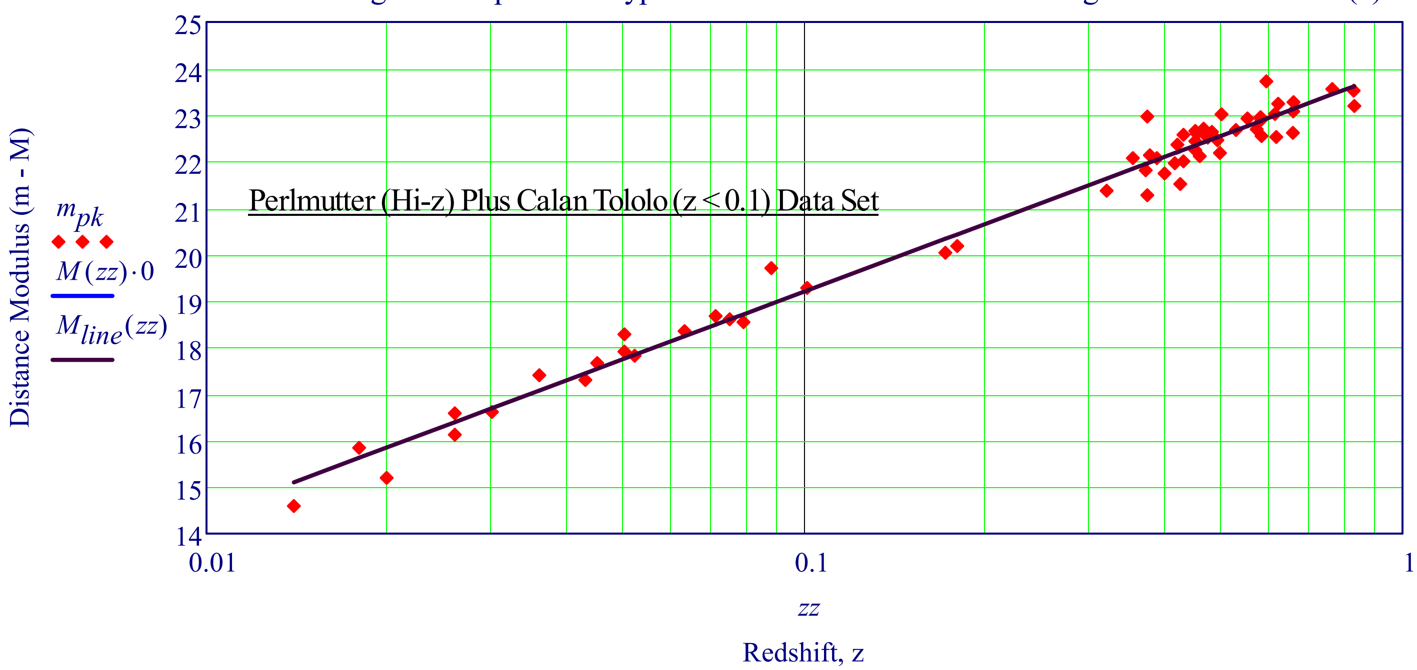
Fit a Logfit Function and Straight Line to Magnitude vs. Redshift Data

$zz := ZD^{\langle 0 \rangle} \qquad max(zz) = 0.83 \qquad m_{pk} := ZD^{\langle 2 \rangle} \qquad max(m_{pk}) = 23.73$

$ab := logfit(zz, m_{pk}, vg) \qquad M(z) := ab_0 \cdot ln(z + ab_1) + ab_2$

$ba := line(log(zz), m_{pk}) \qquad M_{line}(z) := ba_0 + ba_1 \cdot log(z) \qquad ba_1 = 4.803$

Hubble Diagram: Supernova Type 1a Measurement - Effective Magnitude vs. Redshift (z)



Find the Percent of z > 0.1 Supernovae that are above the Regression Line, Mline

$$PercentAboveMean := \left(\sum_{n=30}^{59} if\left(m_{pk_n} - M_{line}(zz_n) > 0, 1, 0\right) \right) \frac{1}{30}$$

$PercentAboveMean = 56.667 \cdot \%$

This shows that the Velocities of the High z Galaxies are statistically increasing faster than the mean Hubble Constant. **The Expansion is Accelerating.**

XXVIIC. THE 5 Year DARK ENERGY SURVEY AND ITS SUPERNOVAE - 2024

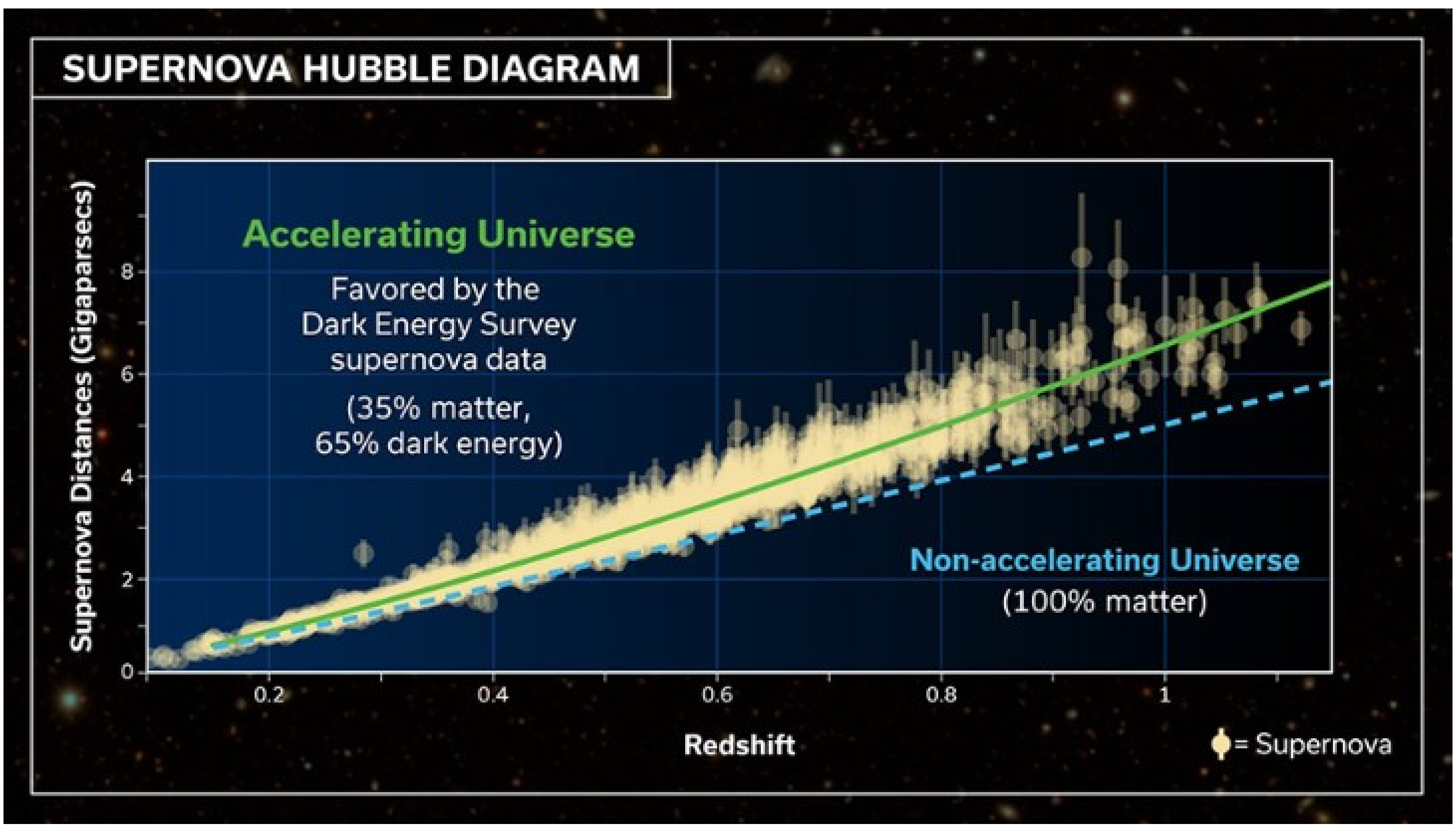
Refer to the Article:
The Dark Energy Survey (DES): Cosmology Results With ~1500 New High-redshift Type Ia Supernovae Using The Full 5-year Dataset January 9, 2024 https://arxiv.org/abs/2401.02929

https://skyandtelescope.org/astronomy-news/cosmology/how-strong-is-dark-energy-intriguing-findings-from-new-supernova-catalog/

We have known for nearly 100 years that the universe is expanding. But only at the turn of the 21st century did astronomers discover that the expansion was actually speeding up.
Now, this new study suggests ***that this phenomenon might be weaker than we thought.***
The Previous value for Λ was 69%. This DES Study gives Λ = 65%. See Plot Below.

The largest sample of Type Ia supernovae ever made by a single telescope sheds light on dark energy.

The Dark Energy Survey (DES) was conceived to characterize the properties of dark matter and dark energy with **unprecedented precision and accuracy** through **four primary observational probes** (The Dark Energy Survey Collaboration 2005; Bernstein et al. 2012; Dark Energy Survey Collaboration 2016; Lahav et al.2020).
An example of a supernova discovered by the Dark Energy Survey (DES) within the field covered by one of the individual detectors in the Dark Energy Camera. The supernova exploded in a spiral galaxy with redshift = 0.04528, which corresponds to a light-travel time of about 0.6 billion years. This is one of the nearest supernovae in the sample. In the inset, the supernova is a small dot at the upper-right of the bright galaxy center. ***DES collaboration***



During a five-year survey, astronomers used a special camera mounted on the Víctor M. Blanco 4-meter Telescope at Cerro Tololo Inter-American Observatory to discover **1,635 Type Ia supernovae from hundreds of different galaxies** spread over a huge range of distances. The light from these supernovae is anywhere between 1 billion and 9 billion years old. Using the aforementioned standard-candle technique, the team calculated the universe's expansion rate — and **established the first good constraints on dark energy.**

XXVIID. Compare the Theoretical Magnitude-Redshift to Perlmutter 1999 SB 1A

Theoretical Apparent Magnitude-Redshift Relation (Mukhanov)

Physical Foundations of Cosmology, Mukhanov, Equations 2.78 and 2.81

$$\chi_{em}(z, \Omega_m) := \int_0^z \frac{1}{\sqrt{\Omega_m \cdot (1 + z\xi)^3 + (1 - \Omega_m)}} \, dz\xi$$

$$\Phi^2(\chi_{em}) = \begin{cases} \sinh^2 \chi, & k = -1; \\ \chi^2, & k = 0; \\ \sin^2 \chi, & k = +1. \end{cases}$$

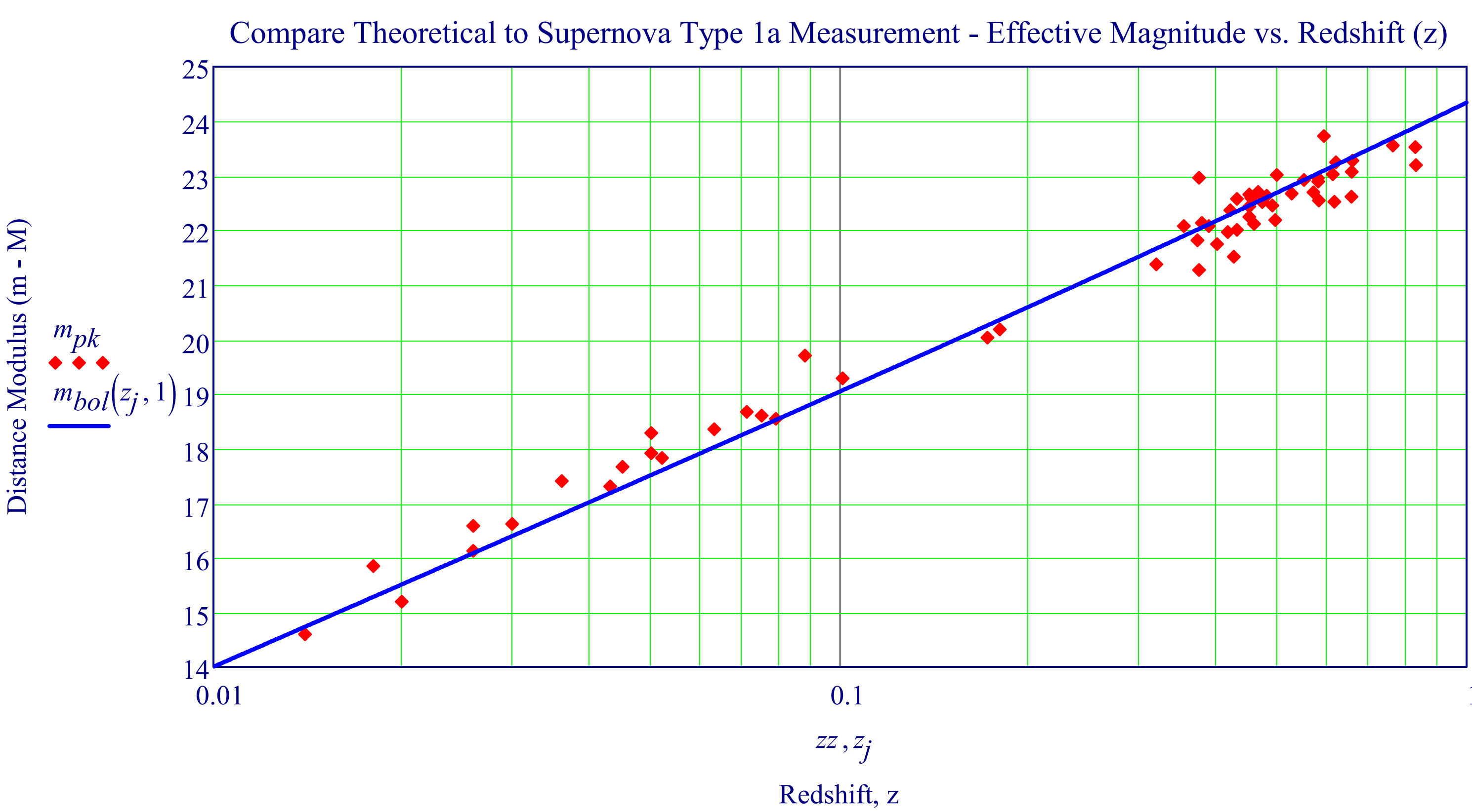
Note: For k = 0

$$\Phi(\chi_{em}) = \chi_{em}$$

Then the Theoretical Bolometric Magnitude for k = 0 is Given by:

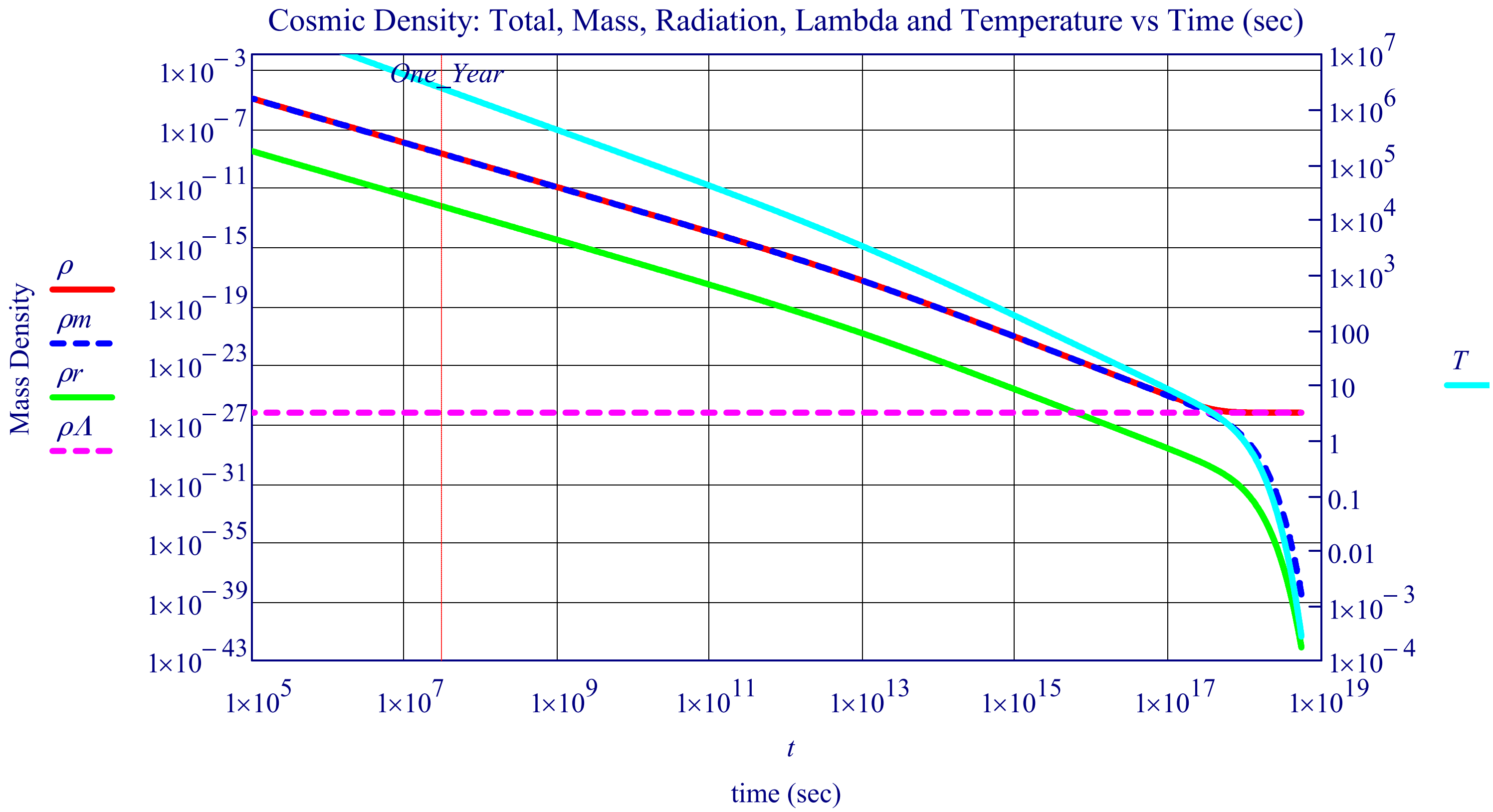
$$m_{bol}(z, \Omega_m) := 5\,log(1 + z) + 5\,log(\chi_{em}(z, \Omega_m)) + 24$$

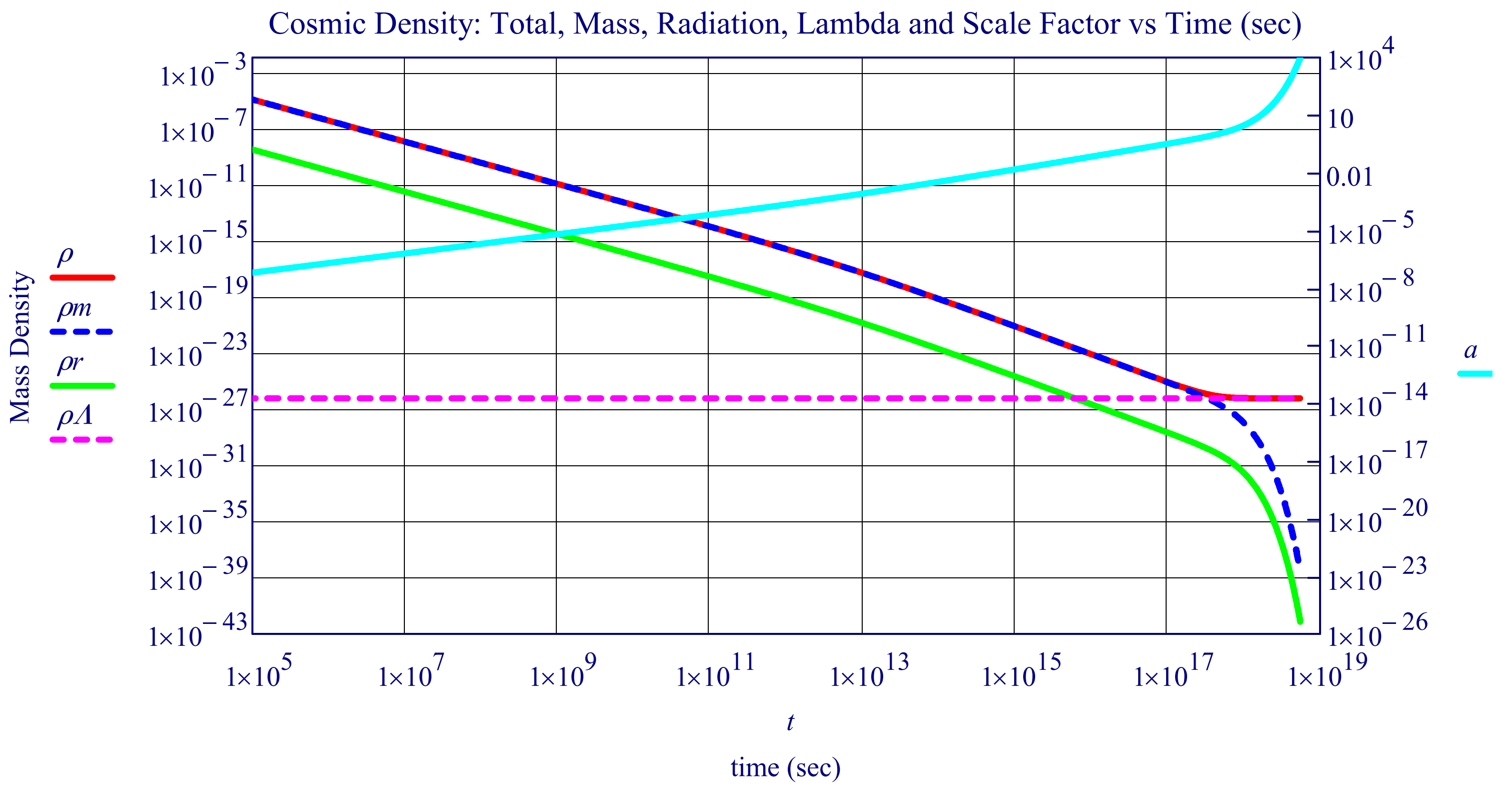
$$j := 0 .. 300 \qquad z_j := 10^{0.01 \cdot j - 3}$$

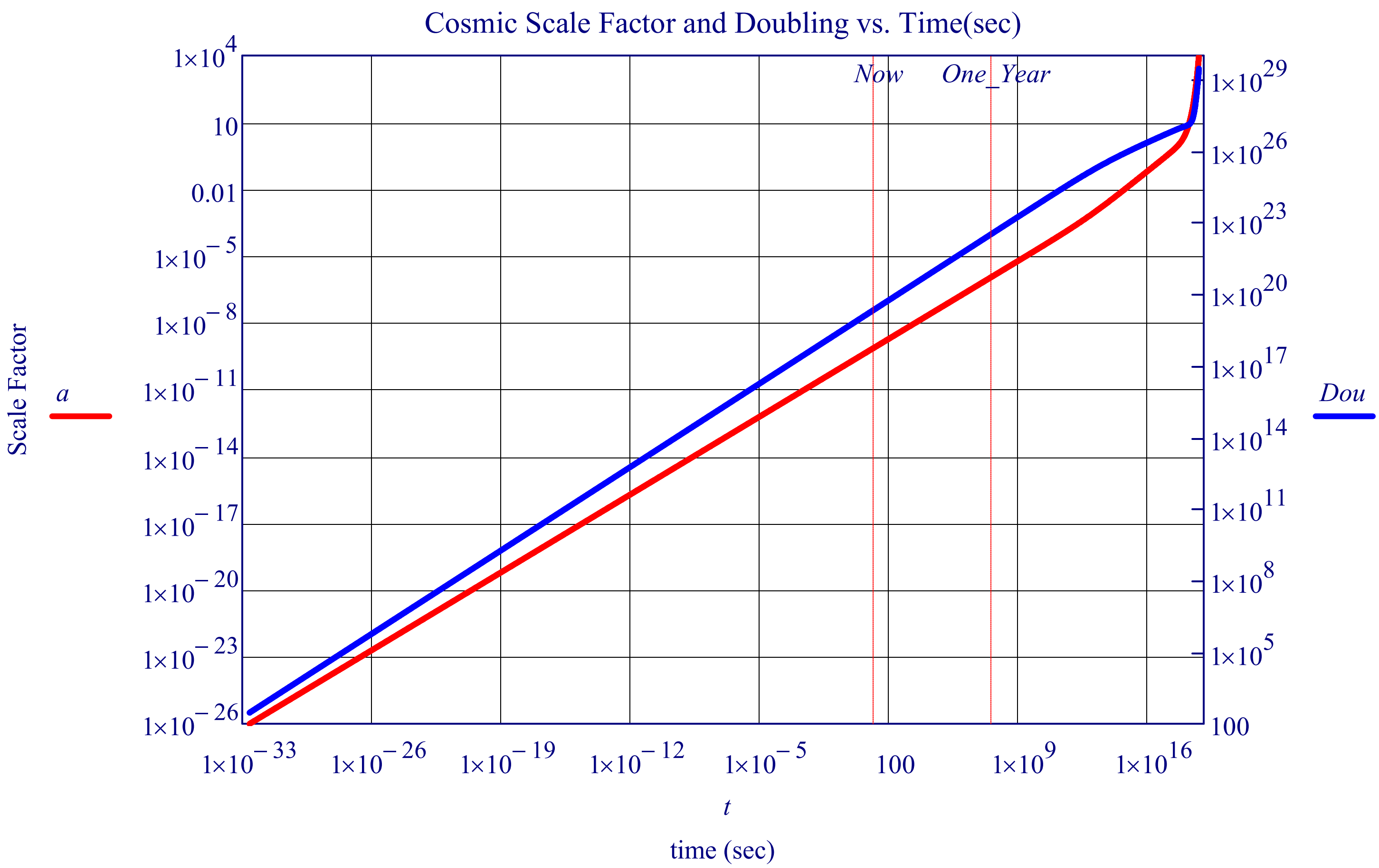


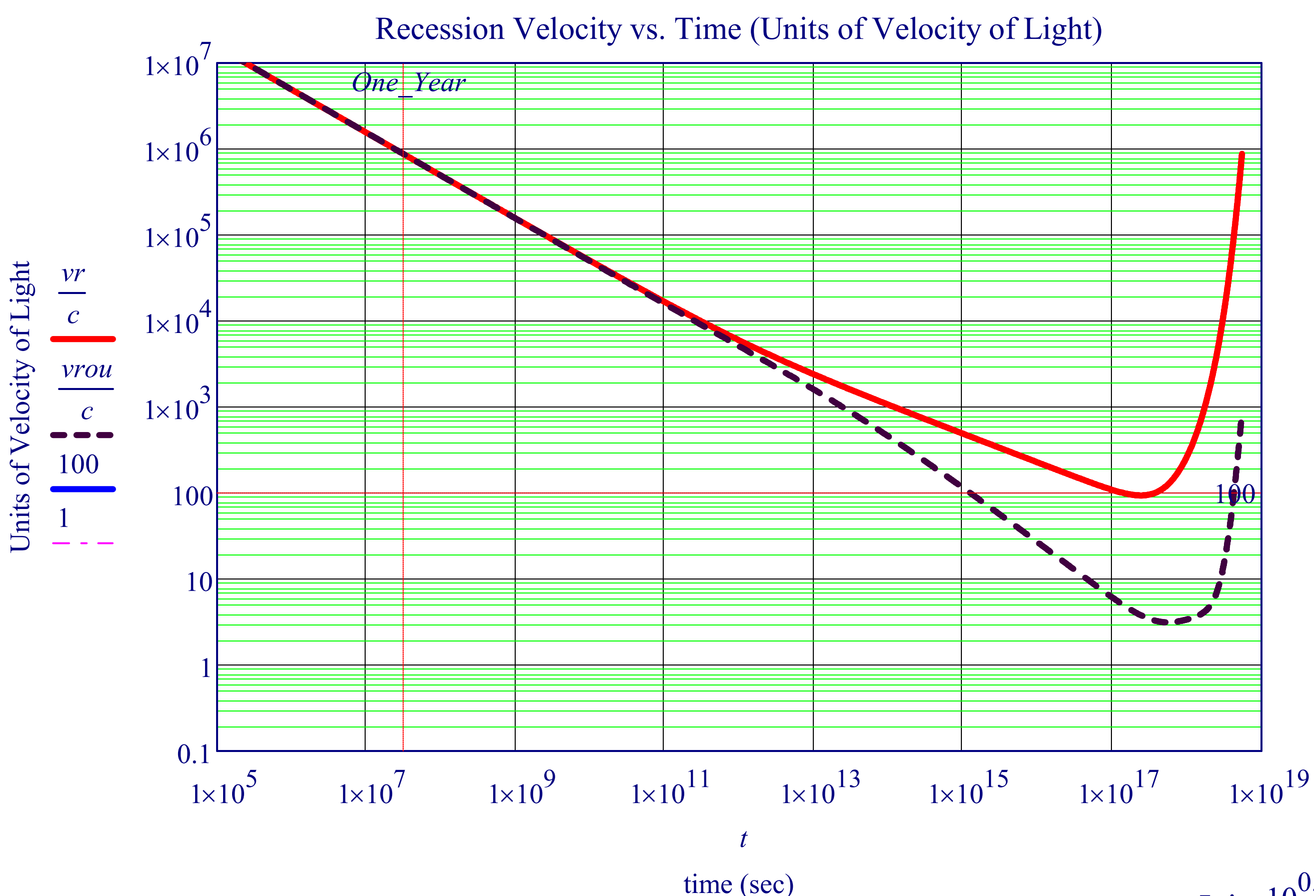
XXVIII. Exploring the Behavior of Some Cosmology Models by Plotting Their Parameters Given by the Definitions in Section VII.

Plots of Cosmic Density Components, Scale Factor, Recession Velocity, Hubble Factor Cosmic Scale Factor, Components of the Energy of the Universe

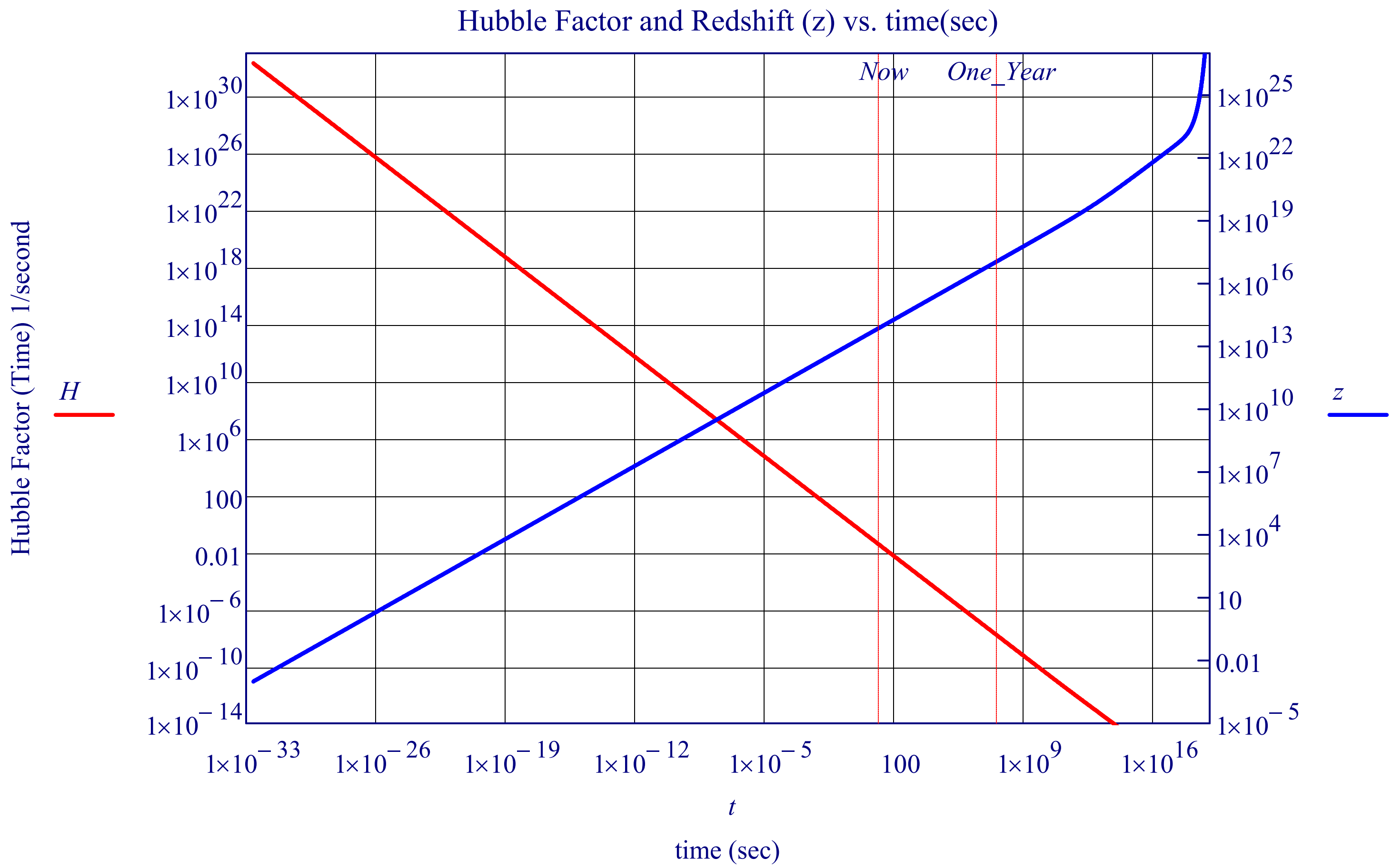




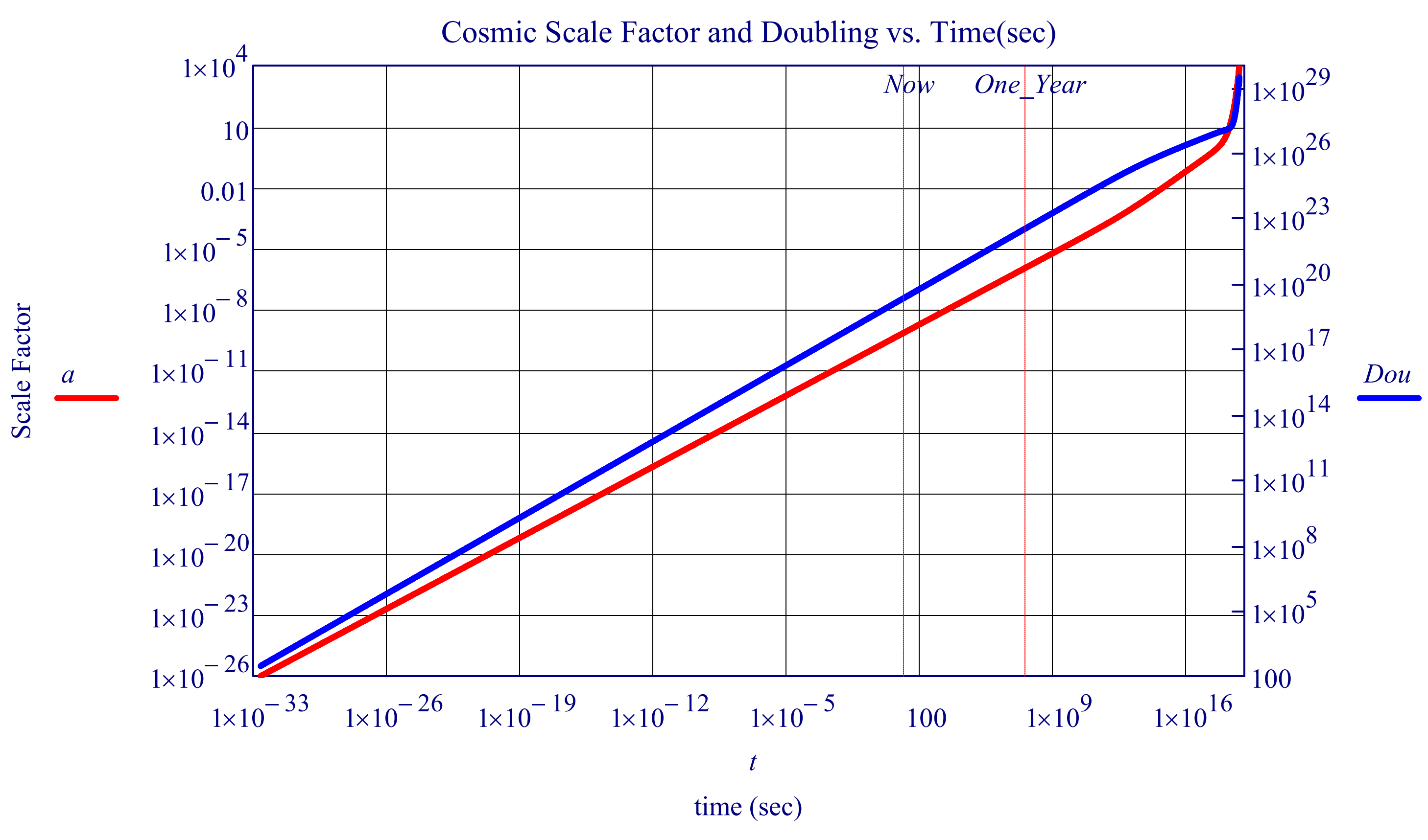




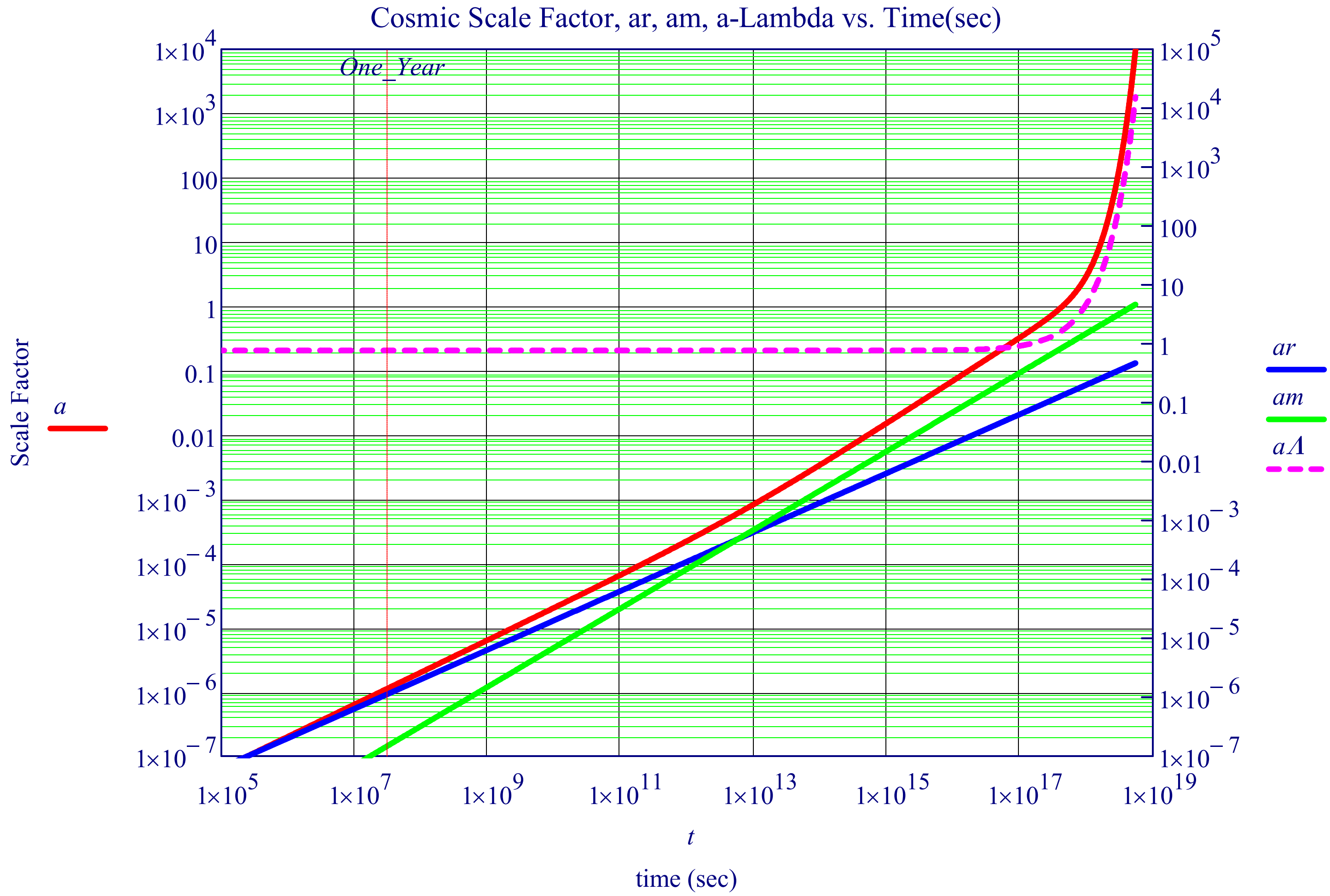
$z_i := 10^{0.01 \cdot i - 3}$

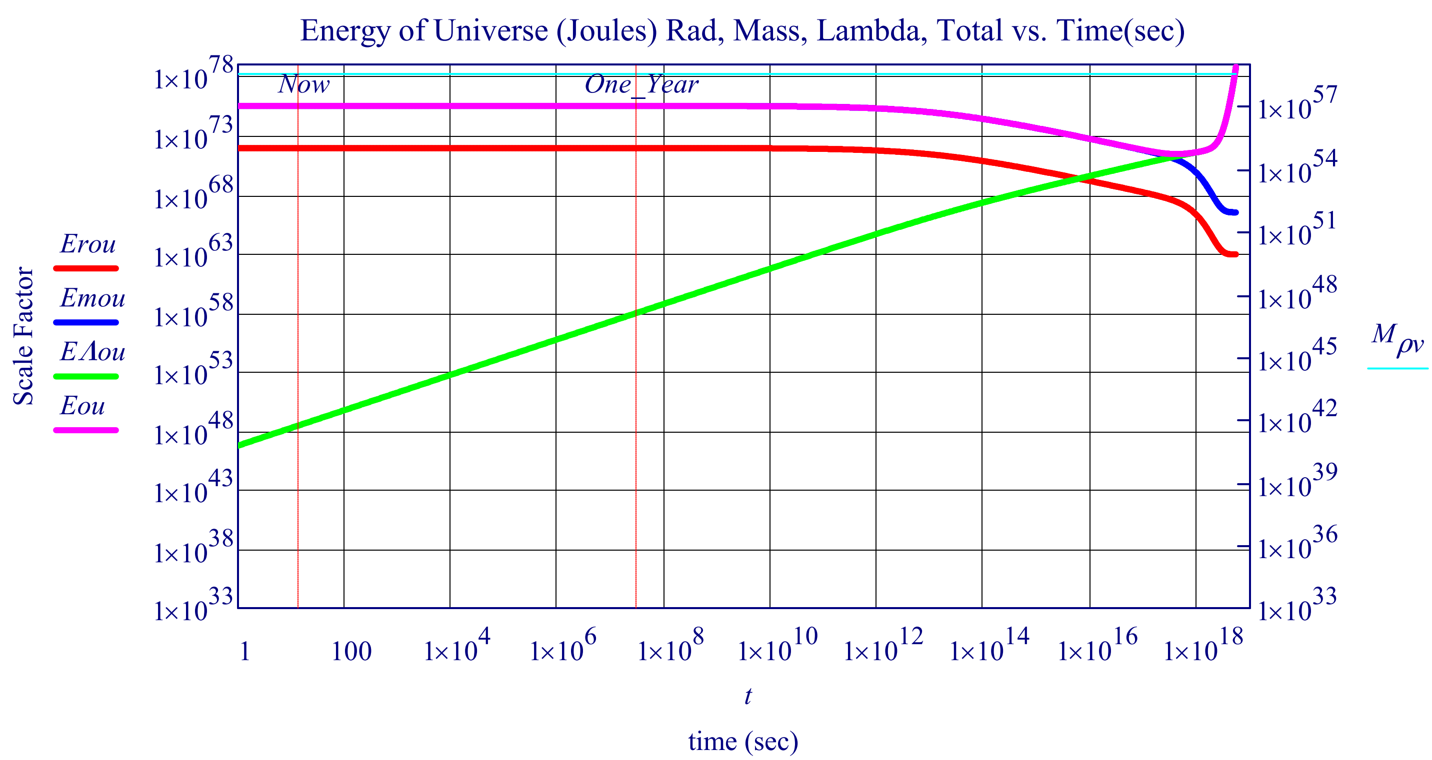


Hubble Factor and Redshift (z) vs. time(sec)
Now
One_Year
Hubble Factor (Time) 1/second
H
z
time (sec)
t



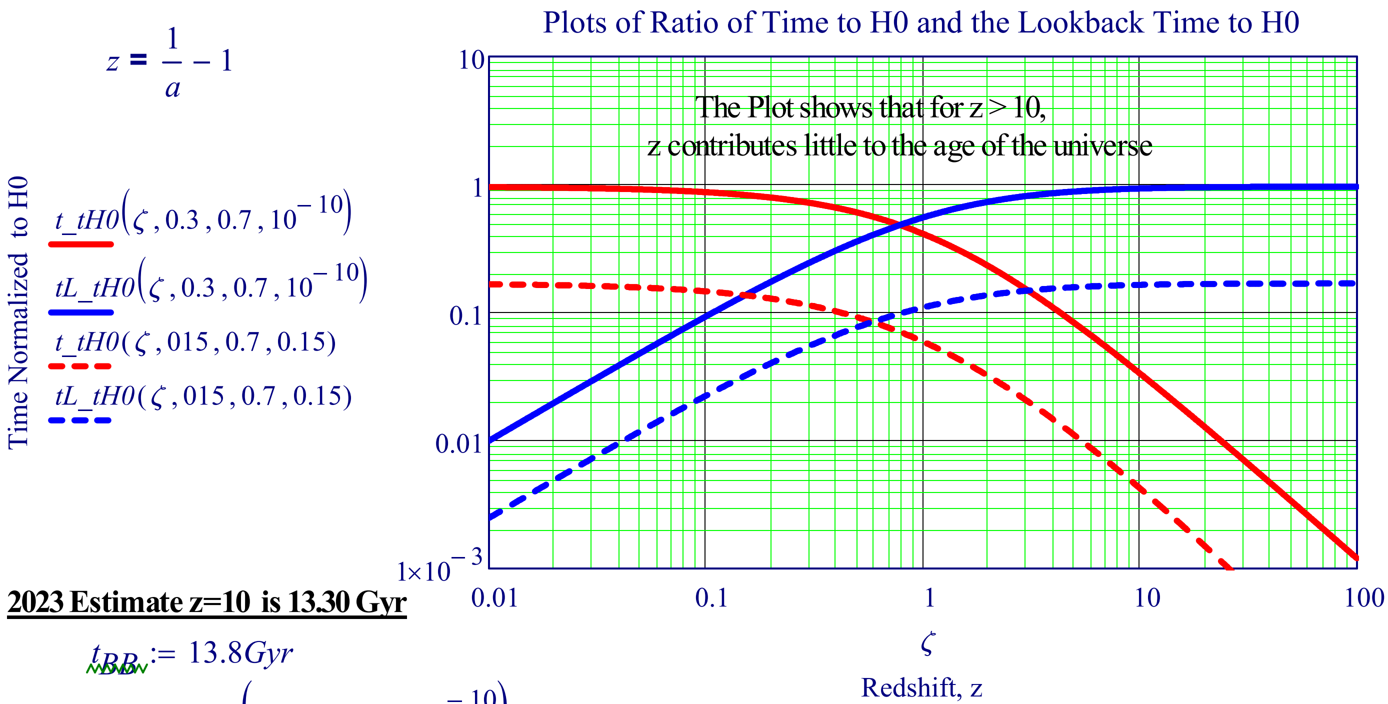
Cosmic Scale Factor and Doubling vs. Time(sec)
Now
One_Year
Scale Factor
a
Dou
time (sec)
t


Cosmic Scale Factor, ar, am, a-Lambda vs. Time(sec)
One_Year
Scale Factor
a
ar
am
aΛ
t
time (sec)


Energy of Universe (Joules) Rad, Mass, Lambda, Total vs. Time(sec)
Now
One_Year
Scale Factor
Erou
Emou
EΛou
Eou
Mρv
t
time (sec)

XXIX. Lookback Time versus Red Shift and Age of Universe

$z = \frac{1}{a} - 1$



2023 Estimate z=10 is 13.30 Gyr

$t_{BB} := 13.8Gyr$

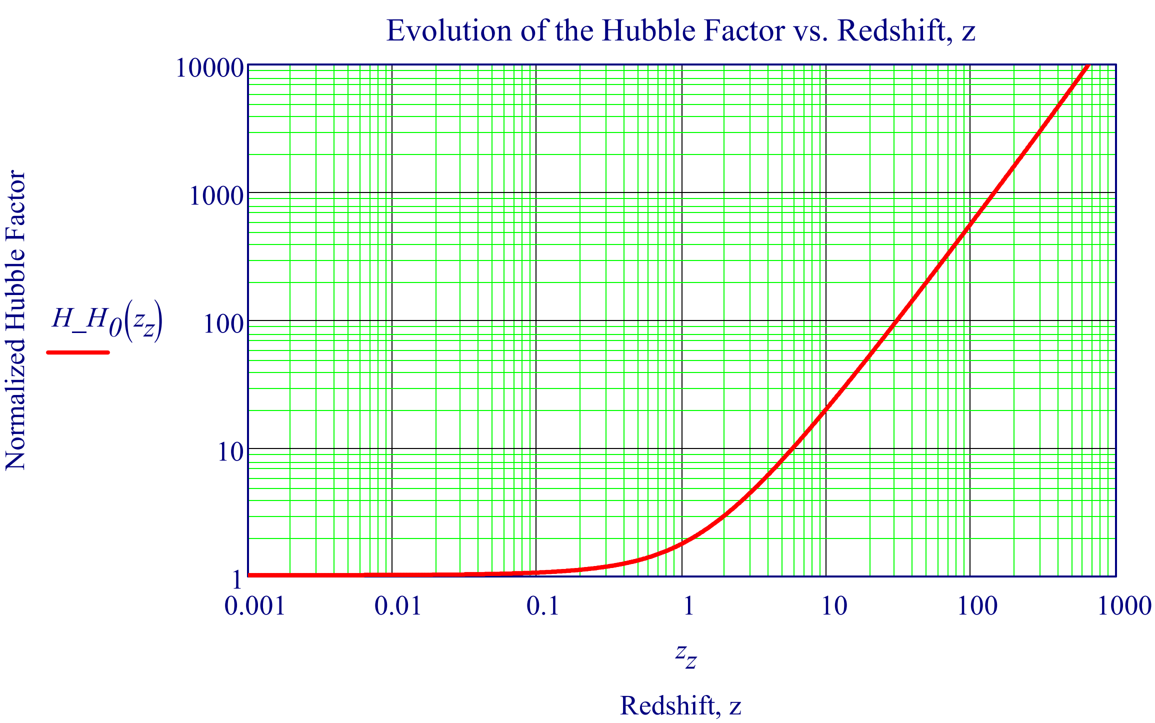
$t_{BB} \cdot tL\_tH0(10, 0.3, 0.7, 10^{-10}) = 12.844 \cdot Gyr$

Evolution of the Hubble Factor: Mass Conservation of non-relativistic matter implies $\rho_m \propto a^{-3} = (1+z)^3$. In the ΛCDM model, dark energy is assumed to behave like a cosmological constant: $\rho_\Lambda \propto a^0 = (1+z)^0$. The density of radiation (and massless neutrinos) scales as $\rho_r \propto a^{-4} = (1+z)^4$ because the number density of photons is $\propto a^{-3} = (1+z)^3$ and the mass $E/c^2 = h\nu/c^2$ of each photon scales as $E \propto \lambda^{-1} \propto (1+z)^1 \propto a^{-1}$.

Dynamical Equation Specifying the Evolution of the Hubble Factor of Our Universe

$\Omega_{r0} := 8.7 \times 10^{-5}$

$$\frac{H}{H_0} = H\_H_0(z) := \sqrt{\Omega_{m0} \cdot (1+z)^3 + \Omega_{\Lambda 0} + \Omega_{r0} \cdot (1+z)^4}$$



XXX. Nucelosynthesis in the Early Universe:

Modeling Hydrogen Orbitals (We will use the Maple Programming Language for Model)

$Y(l, m, \theta, \phi)$ is the spherical harmonic or angular part of an orbital, where l is the angular momentum (azimuthal) quantum number and m is the magnetic quantum number. θ is the angle with the z axis in spherical coordinates and ϕ is the angle around the z axis in spherical coordinates. These angles follow the quantum mechanics convention, used here and in the VectorCalculus package, which is different from the math convention used in the rest of Maple.

A d_{z^2} orbital has $l = 2$ and $m = 0$ and an angular part that is the spherical harmonic $Y(2, 0, \theta, \phi)$. (This worksheet makes liberal use of atomic variables to make nice looking variables such as d_{z^2}, check "Atomic Variables" in the view menu to highlight these.)
The function *cartesian* converts the spherical harmonic to the usual form in terms of x, y, z and r (use the function *fullcartesian* to remove the last r)

> $d_{z^2} := Y(2, 0, \theta, \phi);$

$$d_{z^2} := \frac{1}{4} \frac{\sqrt{5}\,\left(3 \cos(\theta)^2 - 1\right)}{\sqrt{\pi}}$$

$$-\frac{1}{4} \frac{\sqrt{5}\,\left(x^2 + y^2 - 2 z^2\right)}{r^2 \sqrt{\pi}}$$

The plots of orbitals usually seen are just plots of the squares of their angular parts (for contour plots of the wavefunction with both radial and angular parts, see below). Recall again that Maple's (θ, ϕ) is (ϕ, θ) in quantum mechanics, so put ϕ before θ in the plot command. A useful way to color these is by phase. Since this spherical harmonic is real, the phase simply indicates the sign: red for positive (phase = 0), cyan for negative (phase = π).

> $plot3d\left(d_{z^2}^{\,2}, \phi = 0 ..2\,\pi, \theta = 0 ..\pi, coords = spherical, style = patchnogrid, scaling = constrained, color = \mathrm{argument}\left(d_{z^2}\right) / (2\,\pi), grid = [50, 50], axes = none\right)$

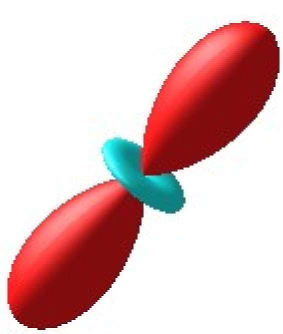
Only the spherical harmonics with $m = 0$ are real. For example, for $l = 2$, $m = 1$ we have

> $d_1 := Y(2, 1, \theta, \phi)$

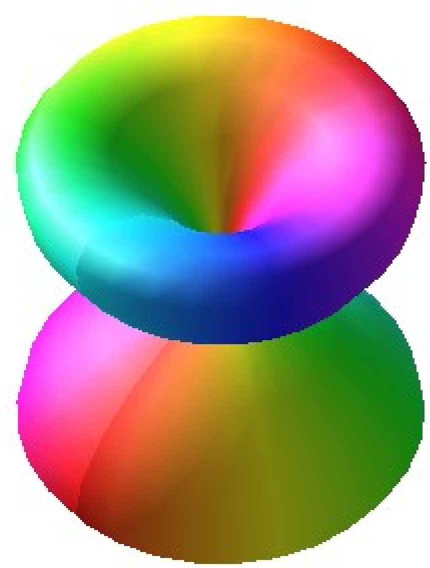
$$d_1 := \frac{1}{4} \frac{\sqrt{30}\,\sin(\theta) \cos(\theta)\, \mathrm{e}^{I\phi}}{\sqrt{\pi}} \tag{3.2}$$

The square of the absolute value can be plotted in the same way as above. The colors now show phases other than 0 and π.

> $plot3d\left(|d_1|^2, \phi = 0 ..2\,\pi, \theta = 0 ..\pi, coords = spherical, style = patchnogrid, scaling = constrained, color = \mathrm{argument}\left(d_1\right) / (2\,\pi), grid = [30, 30], axes = none\right)$



The square of the absolute value can be plotted in the same way as the spherical harmonic at the left. The colors now show phases other than 0 and π.



Nucelosynthesis in the Early Universe: Ratio of Neutrons to Protons

The basic building blocks for nucleosynthesis are neutrons and protons. As the Universe cools, protons and neutrons become stable particles and they, in turn, bind into nuclei. With a decay time of only fifteen minutes, the existence of a free neutron is as fleeting as fame; once the universe was several hours old, it contained essentially no free neutrons. However, a neutron which is bound into a stable atomic nucleus is preserved against decay. There are still neutrons around today, because they've been tied up in deuterium, helium, and other atoms.

The Boltzmann distribution for the number density of nonrelativistic nuclei of atomic weight A is: $n_A \sim T^{3/2}\, e^{(\mu A - m A)/k}$.

Given the masses of the particles in Mega Electron Volts (MeV), the number density for neutrons and protons is:

$MeV := 1.60218 \times 10^{-13} \cdot J \qquad m_n := 939.565420 MeV \quad m_p := 938.272088 MeV \qquad m_n - m_p = 1.293 \cdot MeV$

$$n_n = g_n \cdot \left(\frac{m_n \cdot k \cdot T}{2 \pi h_{bar}^{\,2}} \right)^{\frac{3}{2}} \cdot e^{\frac{-m_n \cdot c^2}{k_b \cdot T}} \qquad n_p = g_p \cdot \left(\frac{m_p \cdot k \cdot T}{2 \pi h_{bar}^{\,2}} \right)^{\frac{3}{2}} \cdot e^{\frac{-m_p \cdot c^2}{k_b \cdot T}}$$

Since the statistical weights of protons and neutrons are equal, with $g_p = g_n = 2$, the neutron-to-proton ratio is then given by the equation:

$$Ratio_{n\_p}(T) := \left(\frac{m_n}{m_p} \right)^{\frac{3}{2}} \cdot e^{-(m_n - m_p) \cdot \frac{c_v^{\,2}}{k_b \cdot T \cdot K}}$$

These reactions continued until the decreasing temperature and density caused the reactions to become too slow, which occurred at about T=0.7 MeV (time around 1 second) and is called the freeze out temperature.

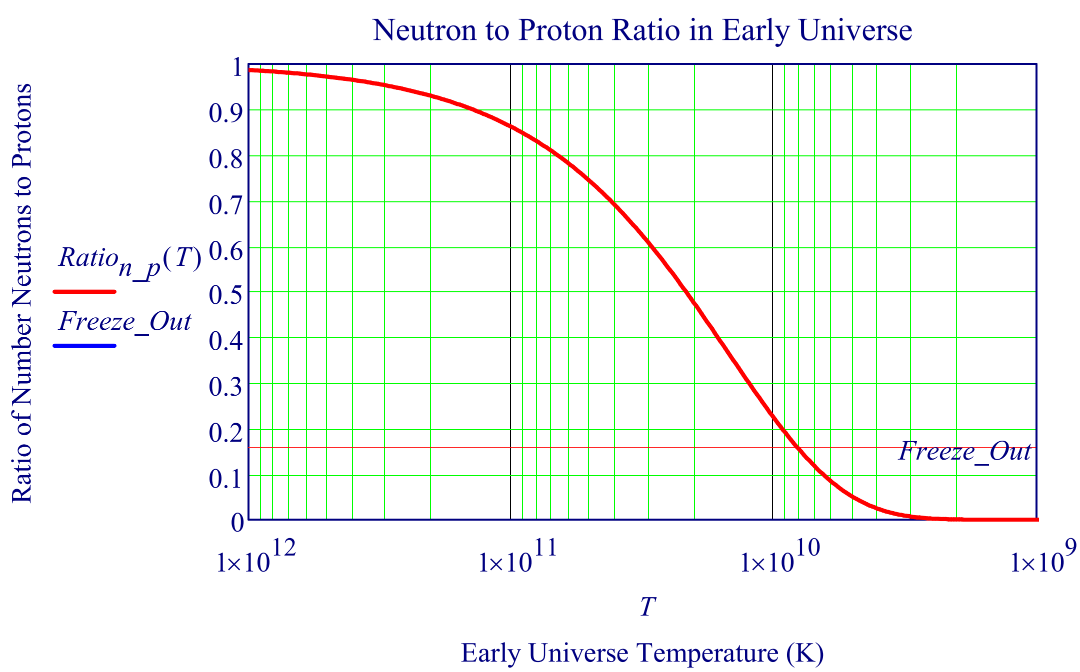
Freeze Out Temperature in Kelvin, K

$$T_{Freeze\_Out} := \frac{0.7 \cdot MeV}{k_b \cdot K} = 8.123 \times 10^9$$

Ratio of Neutrons to Protons in Early Universe

$$Freeze\_Out := Ratio_{n\_p}\left(T_{Freeze\_Out}\right) = 0.158$$

$n := 1 .. 1000 \qquad T_n := 1000000000000 - n$



XXXI. Λ-CDM Model Parameters

http://astro.vaporia.com/start/lambdacdm.html

Wikipedia

The current standard model of cosmology, the Lambda-CDM model, uses the FLRW metric. By combining the observation data from some experiments such as WMAP and Planck with theoretical results of Ehlers–Geren–Sachs theorem and its generalization, astrophysicists now agree that the early universe is almost homogeneous and isotropic (when averaged over a very large scale) and thus nearly a FLRW spacetime. That being said, attempts to confirm the purely kinematic interpretation of the Cosmic Microwave Background (CMB) dipole through studies of radio galaxies and quasars show disagreement in the magnitude. Taken at face value, these observations are at odds with the Universe being described by the FLRW metric. Moreover, one can argue that there is a maximum value to the Hubble constant within an FLRW cosmology tolerated by current observations, $H_0 = 71 \pm 1$ km/s/Mpc, and depending on how local determinations converge, this may point to a breakdown of the FLRW metric in the late universe, necessitating an explanation beyond the FLRW metric

parameter	symbol	determined value
physical baryon density	$\Omega_b h^2$	0.02230
physical dark matter density	$\Omega_d h^2$	0.1188
age of the universe	t_0	13.799×10^9 years
scalar spectral index	n_s	0.9667
curvature fluctuation amplitude	Δ^2_R	2.441×10^{-9}
optical depth to the epoch of reionization (EOR)	τ	0.066

item	symbol	calculated value
Hubble constant	H_0	67.74 km $s^{-1}Mpc^{-1}$
baryon density parameter	Ω_b	0.0486
dark matter density parameter	Ω_d	0.2589
matter density parameter	Ω_m	0.3089
dark energy density parameter	Ω_Λ	0.6911

Parameters for Galactic mass components	
Component, Parameter, Value, Uncertainty	
Bulge Mass M b = 1.80 × 10 10 M ⊙ ~ 5 %	**Bar for 3 kpc dip** Amplitude δ bar > 0.8 × Σ d —
Half-mass scale radius R b = 0.5 kpc	Assumed bar half length † 1.7 kpc —
SMD at R b Σ be = 3.2 × 10 3 M ⊙ pc −2	Assumed tilt angle † 13 ◦ —
Center SMD Σ bc = 6.8 × 10 6 M ⊙ pc −2	**Bulge, disk, rings** Total mass M bdr = 8.3 × 10 10 M ⊙ ~ 5 %
Center volume density ρ bc = ∞ —	**Dark halo**
Disk Mass M d = 6.5 × 10 10 ~ 5 %	Mass in r = 10kpc sphere M h (10kpc) = 4.2 × 10 10 M ⊙ ~ 10 %
Scale radius R d = 3.5 kpc	(Sphere) Mass in r = 20 kpc sphere ‡ M h (20kpc) = 1.24 × 10 11 M ⊙
Center SMD Σ dc = 8.44 × 10 2 M ⊙ pc −2	Core radius R h = 5.5 kpc
Center volume density ρ dc = 8M ⊙ pc −3	Central SMD in | z | < 10 kpc Σ hc = 352M ⊙ pc −2
Rings Mass M r ~ 0	Central volume density ρ hc = 0.03M ⊙ pc −3
Peak Σ r 0.17 and 0.34 × Σ d ~ 20 %	Circular velocity at infinity V ∞ = 200km s-1
Radii of wave nodes R r = 3 and 9.5 kpc ~ 3	**Total Galactic mass**
Widths w r = 1 and 2 kpc ~ 10	Mass in r = 20 kpc sphere M total (20kpc) = 2.04 × 10 11 M ⊙ ~ 10 %

XXXII. The Inflation Hypothesis and the Very Early Universe

A hypothesis is an educated guess or prediction about the relationship between two variables. It must be a testable statement; something that you can support or falsify with observable evidence. The objective of a hypothesis is for an idea to be tested, not proven.

The standard Hot Big Bang model, in which the early universe was radiation- dominated, is not without its flaws. In particular, after the discovery of the cosmic microwave background led to the widespread embrace of the Big Bang, it was realized that the standard Hot Big Bang scenario had three underlying problems. These nagging problems were called the flatness problem, the horizon problem, and the monopole problem. The flatness problem can be summarized by the statement, "The universe is nearly flat today, and was even flatter in the past." The horizon problem can be summarized by the statement, "The universe is nearly isotropic and homogeneous today, and was even more so in the past." The monopole problem can be summarized by the statement, "The universe is apparently free of magnetic monopoles."

What is the concept of inflation? In a cosmological context, inflation can most generally be defined as the hypothesis that there was a period, early in the history of our universe, when the expansion was accelerating outward; that is, an epoch when the acceleration equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P), \qquad \frac{d}{dt}a = \sqrt{\frac{8\pi G\cdot\rho_\nu}{3}}\cdot a = H_\nu\cdot a$$

tells us that when $P < -\varepsilon/3$. Thus, inflation would have taken place if the universe were temporarily dominated by a component with equation-of-state parameter $w < -1/3$. The simplest implementation of inflation states that the universe was temporarily dominated by a positive cosmological constant Λ_i (with $w = -1$), and thus had an acceleration equation that could be written in the form

$$\frac{\ddot{a}}{a} = \frac{\Lambda_i}{3} > 0$$

the Hubble constant H_i during the inflationary phase was thus constant, with the value $H_i = (\Lambda_i/3)^{1/2}$, and the scale factor grew exponentially with time: $a(t) \propto e^{H_i t}$.

During inflation, the universe is dominated by the vacuum energy. In a time interval, Δt the universe expands by a factor $\exp(H_\nu \Delta t)$. Define the Doubling Time, t_D, as the time it takes the universe to double in size.

During inflation, the universe is dominated by vacuum energy. In the early universe, when the scale factor is very small, then mass density ρ_m must be much greater than ρ_ν. Matter density ρ_m is diluted. Then Doubling Time, t_D, is:

$$\rho_\nu := 10^{71}\frac{gm}{cm^3} \qquad H_\nu := \sqrt{\frac{8\pi\cdot G\cdot\rho_\nu}{3}}$$

$$e^{H_\nu\cdot t_D} = 2 \qquad t_D := H_\nu^{-1}\cdot log(2,e)$$

$$t_D = 2.932\,s\cdot 10^{-33}$$

To see how a period of exponential growth can resolve the flatness, horizon, and monopole problems, suppose that the universe had a period of exponential expansion sometime in the midst of its early, radiation-dominated phase. For simplicity, suppose the exponential growth was switched on instantaneously at a time t_i, and lasted until some later time t_f, when the exponential growth was switched off instantaneously, and the universe reverted to its former state of radiation-dominated expansion. In this simple case, we can write the scale factor as

$$a(t) = \begin{cases} a_i(t/t_i)^{1/2} & t < t_i \\ a_i e^{H_i(t-t_i)} & t_i < t < t_f \\ a_i e^{H_i(t_f-t_i)}(t/t_f)^{1/2} & t > t_f. \end{cases}$$

Thus, between the time t_i, when the hypothesized exponential inflation began, and the time t_f, when the inflation stopped, the scale factor increased by a factor

$$\frac{a(t_f)}{a(t_i)} = e^N$$

where N, the number of e-foldings of inflation, would be

$$N \equiv H_i(t_f - t_i)$$

If the duration of inflation, $t_f - t_i$, was long compared to the Hubble time during inflation, then N was large, and the growth in scale factor during a hypothetical inflationary period would be enormous.

For concreteness, let's take one possible model for inflation. This model states that exponential inflation started around the GUT time, $t_i \approx t_{GUT} \approx 10^{-36}$ s, with a Hubble parameter and lasted for N e- foldings, ending at $t_f \approx (N+1)t_{GUT}$. Note that the cosmological constant Λ_i present at the time of inflation in this model was very large compared to the cosmological constant that is present today. Currently, the evidence is consistent with an energy density in Λ of $\varepsilon_{\Lambda,0} \approx 0.69\varepsilon_{c,0} \approx 0.0034$ TeVm^{-3}. To produce exponential expansion with a Hubble parameter Hi $\approx 10^{36}$ s^{-1}, the cosmological constant during inflation would have had an energy density

$$\varepsilon_{\Lambda_i} = \frac{c^2}{8\pi G}\Lambda_i = \frac{3c^2}{8\pi G}H_i^2 \sim 10^{105}\,\text{TeV m}^{-3},$$

over 10^7 orders of magnitude larger.

Prior to the inflationary period, the universe was radiation-dominated. Thus, the horizon distance at the beginning of inflation was

$$d_{\text{hor}}(t_i) = a_i c \int_0^{t_i} \frac{dt}{a_i(t/t_i)^{1/2}} = 2ct_i.$$

The horizon size at the end of inflation was

$$d_{\text{hor}}(t_f) = a_i e^N c \left(\int_0^{t_i} \frac{dt}{a_i(t/t_i)^{1/2}} + \int_{t_i}^{t_f} \frac{dt}{a_i \exp[H_i(t-t_i)]} \right)$$

If N, the number of e-foldings of inflation, is large, then the horizon size at the end of inflation was

$$d_{\text{hor}}(t_f) = e^N c(2t_i + H_i^{-1})$$

An epoch of exponential inflation causes the horizon size to grow exponentially. If inflation started at $t_i \approx 10^{-36}$s, then the horizon size immediately

$$d_{\text{hor}}(t_f) = a_i e^N c \left(\int_0^{t_i} \frac{dt}{a_i(t/t_i)^{1/2}} + \int_{t_i}^{t_f} \frac{dt}{a_i \exp[H_i(t-t_i)]} \right)$$

$$d_{\text{hor}}(t_i) = 2ct_i \approx 6 \times 10^{-28}\,\text{m}.$$

For concreteness, let's assume N = 65 e-foldings of inflation, just a bit more than the minimum of 60 e-foldings required to explain the flatness of today's universe. In this fairly minimal model, if we take the horizon size immediately after inflation was

$$d_{\text{hor}}(t_f) \approx e^N 3ct_i \sim 15\,\text{n}$$

During the brief period of ~ 10^{-34}s that inflation lasts in this model, the horizon size is boosted exponentially from submicroscopic scales to something the size of a whale. The exponential increase in the horizon size during inflation is illustrated by the solid line in Figure 10.3. In the post-inflation era, when the universe reverts to being radiation-dominated, the horizon size grows at the rate dhor ∝ a ∝ $t^{1/2}$, as points that were separated by a distance $d_{hor}(t_f)$ at the end of inflation continue to be carried apart from each other by the expansion of the universe.

In the hypothetical model we've adopted, where inflation started around the GUT time and lasted for N = 65 e-foldings, the scale factor was $a(t_f) \sim 2\times10^{-27}$ at the end of inflation, estimated from Equation 10.30. At the time of last scattering, the scale factor was $a(t_{ls}) \approx 1/1090 \approx 9.1 \times 10^{-4}$. Thus, in our model, the horizon distance grew from $dhor(t_f) \sim 15$ m at the end of inflation to $d_{hor}(t_{ls}) \sim 200$ Mpc at the time of last scattering. This is 800 times bigger than the horizon size $d_{hor}(t_{ls}) \approx 0.25$ Mpc that we calculated in the absence of inflation, and is large enough that antipodal points on the last scattering surface are causally connected.

This model states that exponential inflation started around the GUT time, $t_i \approx t_{GUT} \approx 10^{-36}$ s, with a Hubble parameter and lasted for N e- foldings, ending at $t_f \approx (65 + 1)t_{GUT}$.

$$N := 65 \qquad t_i := 10^{-36} s$$

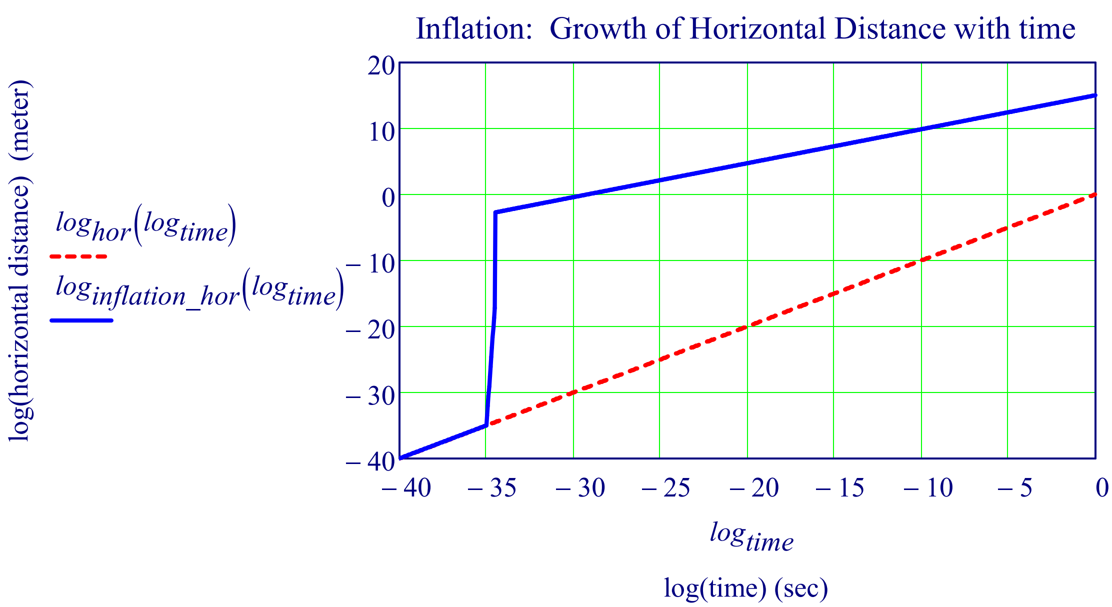
$$t_{GUT} := t_i \qquad t_f := (N + 1)\cdot t_{GUT}$$

$$d_{nor\ tf} := e^N \cdot 3\cdot c\cdot t_i \qquad t_f = 6.6 s\cdot 10^{-35}$$

$$d_{nor\ tf} = 15.254 m$$

$$d_{hor}(t) := 2c\cdot t \qquad d_{hor}(t_i) = 6\cdot 10^{-28} m$$

$$d_{horf}(t) := e^N 3c\cdot t \qquad d_{horf}(t_i) = 15.254 m$$



The solid line shows the growth of the horizon distance in a universe where exponential inflation begins at $t = 10^{-36}$s and lasts for N = 65 e-foldings. The dashed line, for comparison, shows the horizon distance in a radiation-dominated universe without an inflationary epoch.

Biggest Weakness of The Big Bang Theory - the Inflation Hypothesis

The Big Bang Theory starts with the assumption that the universe sprang from a "singularity". Singularity is a mathematical concept and it has no meaning in the realm of Physics. It may be Mathematics, but it certainly is not Physics. It is disturbing that the two main ingredients in ΛCDM, Cold Dark Matter and Dark Energy, are not understood.

The Big Bang Theory is based on the concept of Inflation. Inflation postulates that after 10^{-36} seconds that the universe expanded by a factor of a thousand billion billion billion and then at the right moment the inflation stopped. What is the physical mechanism for inflation? An Inflaton field? How did the inflation know when to stop? How could it have stopped everywhere at the same instant.

In order to explain the rotational velocity of galaxies and a few other phenomena, the concepts of dark matter and dark energy were proposed as explanations. The nature of dark matter is unknown and dark energy is presumed to be the cosmological constant. Quantum theory predict that this constant is 10^{120} times larger than the measured value. This has been referred to as the biggest error ever made in science.
The Big Bang Theory predicts that the initial galaxies that were formed a few millions years after the BB, that galaxies would be formed that would be small in size. Contrary to the predicted, the JWT is finding that there are some large galaxies that were formed at this time.
The Model of GR assumes that the universe is isotropic and homogenous. This may be true locally, but it is not known is this is true in general.

To demonstrate inflation's problems, we will start by following the edict of its proponents: assume inflation to be true without question.

Neil Turok: Physics is in Crisis

Inflation is not a theory. It is a huge collection of models.

During the Planck Era, the symmetry of the matter gets broken due to the curvature of space-time and this is called a trace anomaly. What goes along with this, when you have all these Quantum fields which are describing the matter, so photons, electrons, all of them are associated with a Quantum field. The vacuum field is unable to stand still. The vacuum is not empty. The vacuum consists of all the vibrations of all the fields that you add in the standard model and the problem is those vacuum vibrations should produce huge gravitational waves. “Gravity” detects the energy of the vibrations of particle fields and should produce huge gravitational waves. There have been no primordial gravitational waves detected.

Physicists have essentially been cheating. Taking that vacuum energy of all the fields that we know about and just subtracted it. That is not really consistent. Feynmann acknowledged this. All the great physicists acknowledge this. That what we do is essentially when we do Quantum field Theory and couple it to gravity. This is essentially to cheat.

With Inflation we've found a way around that cheat. We've found a way to cancel the trace anomaly and to cancel the vacuum energy without adding even one particle to the standard model. That mechanism turns out to give fluctuations as a side effect and those fluctuations.
This may match the observations and we then have the best of all possible worlds.

Is the theory at the heart of modern cosmology deeply flawed? **Paul J. Steinhardt**

https://www.scientificamerican.com/article/cosmic-inflation-theory-faces-challenges/

"One thing it would tell us is that at some time shortly after the big bang there had to have been a tiny patch of space filled with an exotic form of energy that triggered a period of rapidly accelerated expansion ("inflation") of the patch. Most familiar forms of energy, such as that contained in matter and radiation, resist and slow the expansion of the universe because of gravitational self-attraction. Inflation requires that the universe be filled with a high density of energy that gravitationally self-repels, thereby enhancing the expansion and causing it to speed up. It is important to note, however, that this critical ingredient, referred to as inflationary energy, is purely hypothetical; we have no direct evidence that it exists. Furthermore, there are literally hundreds of proposals from the past 35 years for what the inflationary energy may be, each generating very different rates of inflation and very different overall amounts of stretching. Thus, it is clear that inflation is not a precise theory but a highly flexible framework that encompasses many possibilities."

Is the theory at the heart of modern cosmology deeply flawed? **Paul J. Steinhardt**

https://www.jstor.org/stable/26002474

Summary:

Highly improbable conditions are required to start inflation. Worse, inflation goes on eternally, producing infinitely many outcomes, so the theory makes no firm observational predictions.

The basic idea of the big bang is that the universe has been slowly expanding and cooling ever since it began some 13.7 billion years ago. This process of expansion and cooling explains many of the detailed features of the universe seen today, but with a catch: the universe had to start off with certain properties.

For instance, it had to be extremely uniform, with only extremely tiny variations in the distribution of matter and energy. Also, the universe had to be geometrically flat, meaning that curves and warps in the fabric of space did not bend the paths of light rays and moving objects. But why should the primordial universe have been so uniform and flat? A priori, these starting conditions seemed unlikely. That is where Guth's idea came in. He argued that even if the universe had started off in total disarray—with a highly nonuniform distribution of energy and a gnarled shape—a spectacular growth spurt would have spread out energy until it was evenly dispersed and straightened out any curves and warps in space.

What gave Guth's idea its appeal was that theorists had already identified many possible sources of such energy. The leading example is a hypothesized relative of the magnetic field known as a scalar field, which, in the particular case of inflation, is known as the "inflaton" field.

The inflaton's potential energy can cause the universe to expand at an accelerated rate. In the process, it can smooth and flatten the universe, provided the field remains on the plateau long enough (about 10^{-30} second) to stretch the universe by a factor of 10^{25} or more along each direction. Inflation ends when the field reaches the end of the plateau and rushes downhill to the energy valley below. At this point, the potential energy converts into more familiar forms of energy—namely, the dark matter, hot ordinary matter and radiation that fill the universe today. The universe enters a period of modest, decelerating expansion during which the material coalesces into cosmic structures.

The self-perpetuating nature of inflation is the direct result of quantum physics combined with accelerated expansion. Recall that quantum fluctuations can slightly delay when inflation ends. Where these fluctuations are small, so are their ef affects. Yet the fluctuations are uncontrollably random. In some re regions of space, they will be large, leading to substantial delays.

Inflating points continue to grow and, in a matter of instants, dwarf the well-behaved region that ended inflation on time. The result is a sea of inflating space surrounding a little island filled with hot matter and radiation. What is more, rogue regions spawn new rogue regions, as well as new islands of matter—each a self-contained universe. The process continues ad infinitum, creating an unbounded number of islands surrounded by ever more inflating space.
What does it mean to say that inflation makes certain predictions—that, for example, the universe is uniform or has scale-invariant fluctuations—if anything that can happen wi happen an infinite number of times?

For inflation, the observed outcome depends sensitively on what the initial state is. That defeats the entire purpose of inflation: to explain the outcome no matter what conditions existed beforehand.

The naive theory supposes that inflation leads to a predictable outcome governed by the laws of classical physics. The truth is that quantum physics rules inflation, and anything that can happen will happen. And if inflationary theory makes no firm predictions, what is its point? The underlying problem is that procrastination carries no penalty—to the contrary, it is positively rewarded. Rogue regions that delay ending inflation continue to grow at an accelerating pace, so they invariably take over.

The Big Bang also leads to the conclusion that most of the matter in the universe is not the "normal" atomic matter with which we are familiar. One of the arguments for the Big Bang is that it appears to be able to account for the relative abundance of the "light" chemical elements such as hydrogen, helium, and lithium. However, the nuclear recipe that accounts for the abundance of these light elements also fixes the total number of protons and neutrons (classified as baryons) generated by the Big Bang. Since atoms contain protons and neutrons, atoms are classified as baryonic matter. Observations suggest the possible existence of large amounts of non-luminous dark matter in addition to the luminous matter (stars and luminous gas) that we can observe. The ratio of total matter to visible matter is often claimed to be roughly ten to one, which implies that dark matter would account for about 90 percent of the matter in the universe. Accounting for this "missing" dark matter is quite difficult, which is why both creationist and evolutionist cosmologists have suggested that what we perceive as large amounts of dark matter may actually result from unknown physics

Cosmology and the Arrow of Time: The Second Law of Thermodynamics- One of Biggest Problems

All the successfull equations of physics are symmetrical in time. Tbey can be used equally well in one direction in time as in the other. The future and the past seem physically to be on a completely equal footing. Newton's Laws, Hamiltons equations, Maxwell's equations, Einstein's general' relativity,'· Dirac's equation, the Schrodinger eqnation . all remain efffectively unaltered if we reverse the direction of time. (Replace the coordinate t which represents time, by -t.) The whole of Classical Physics and part of quantum mechanics is entirely reversible in time.

Our physical understanding actually contains important ingredients other than just equations of time-evolution and some of these do indeed involve time-asymmetries. The most important of these is what is known as the second law of thermodynamics. The low entropy state seems specially ordered, in some manifest way, and the high entropy state, less specially ordered. Define entropy. In rough terms, the entropy of a system is a measure of its manifest disorder. The second law of thermodynamics asserts that the entropy of an isolated system increases with time (or remains constant, for a reversible system).

The concept of phase space or state space is a space in which all possible "states" of a dynamical system or a control system are represented, with each possible state corresponding to one unique point in the phase space. The entropy of a state is a measure of the volume V of the compartment containing the phase-space point which represents the state.

$$\text{Entropy} = k \log V.$$

The number of baryons in the universe is 10^{80}. Now consider the phase space of the entire universe. Each point in the phase space represents a point where there is a different universe. The quantity k is a constant, called Boltzmann's constant. Its value is about 10^{-23} Joules per degree Kelvin. The essential reason for taking a logarithm here is to make the entropy an additive quantity for independent systems.

Putting this together with the Bekenstein-Hawking formula, we find that the entropy of a black hole is proportional to the square of its mass:

$$S_{bh} = m^2 \frac{kG}{h \cdot c}$$

According to a calculation performed in 1929 by Subrahmanyan Chandrasekhar, white dwarfs cannot exist if their masses are more than about 1.4 times the mass of the sun, 1.4 $M_{\odot}$. Note that the Cbandrasekhar limit is not much greater than the sun's mass, whereas many ordinary stars are known whose mass is considerably greater than this value. But there is now a new limit, anaIogus to Chandrasekhar's (referred to as the Landau-Oppenheimer-Volkov limit), whose modem (revised) value is very roughly 2.5 solar masses. The gravitation attraction for a mass greater than this will result in the formation of a black-hole.

Let us consider what was previously thought to supply the largest contribution to the entropy of the universe, namely the 2.7K black-body background radiation. Astrophysicists had been struck by the enormous amounts of entropy that this radiation contains, which is far in excess of the ordinary entropy figures that one encounters in other processes (e.g. in the sun). The background radiation entropy is something like10^8 for every baryon (using natural units, so that Boltzmann's constant, is unity). (In effect, this means that there are 10^8 photons in the background radiation for every baryon.) Thus, with 10^{80} baryons in all, we should have a total entropy of 10^{88}.

The Bekenstein-Hawking formula tells us that the entropy per baryon in a solar mass black hole is about 10^{20} in natmal units so had the universe consisted entirely of solar mass black holes, the total figure would have been very much larger than that given above, namely 10^{100}.

Let us try to be a little more realistic. Rather than populating our galaxies entirely with black holes, let us take them to consist mainly of ordinary stars-some 10^{11} of them and each to have a million (i.e. 10^6) solar-mass black-hole at its core (as might be reasonable for our own Milky Way galaxy). Calculations by Roger Penrose shows that the entropy per baryon would now be actually somewhat larger even than the previous huge figure, namely now 10^{21}, giving a total entropy, in natural umts, of 10^{101}. This figure will give us an estimate of the total phase-space volume V available to the Creator, since this entropy should represent the logarithm of the volume of the (easily) largest compartment. Since 10^{123} is the log of the volume, the volume must be the exponential of 10^{123},

$$V = 10^{10^{123}}$$

XXXIII Proof of the Borde-Guth-Vilenkin (BGV) Theorem

The beginning of-the universe.
The Borde Guth Vilenkin Theorem, indefinitely continued into past., Vilenkin,
*I*nference-review.com/ VOL. 1, NO. 4 / OCTOBER 2015

The BGV theorem demonstrates "that **any inflating model that is globally expanding must be geodesically incomplete in the past**".

Was the big bang truly the beginning of the universe? A beginning in what? Caused by what? And determined by what, or whom? These questions have prompted physicists to make every attempt to avoid a cosmic beginning.

Physicists hoped initially that the singularity might be an artifact of Friedmann's simplifying assumption of perfect uniformity, and that it would disappear in more realistic solutions of Einstein's equations. Roger Penrose closed this loophole in the mid-1960s by showing that, under a very general assumption, the singularity was unavoidable. Under the null convergence condition, gravity always forces light rays to converge.
(Mathematically, t**he null convergence condition (NCC) requires** that the Ricci curvature tensor $R_{\mu\nu}$ must satisfy $R_{\mu\nu}N^{\mu}N^{\nu} \geq 0$ for all null vectors N^{μ}. A null vector is a vector of zero norm, $N\mu N\mu=0$. Combined with Einstein's equations, NCC is equivalent to the null energy condition (NEC), requiring that $T_{\mu\nu}N_{\mu}N_{\nu} \geq 0$ for all null N_{μ}, where $T_{\mu\nu}$ is the Einstein Energy-Momentum Tensor.)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{M_{\rm pl}^2}T_{\mu\nu}$$

Proof

Start with a homogeneous, isotropic, and spatially flat universe with the metric:
This implies that the **density of matter or energy measured by any observer cannot be negative**. The conclusion holds for all familiar forms of classical matter.

$$ds = dt^2 - a^2(t)dx_i dx^i$$

The Hubble expansion rate is $H = \dot{a}/a$, where the dot denotes a derivative with respect to time t. We can imagine that the universe is filled with comoving particles, moving along the timelike geodesics vector x = const. Consider an inertial observer, whose world line is $x_{\mu}(\tau)$, parametrized by the proper time τ. For an observer of mass m, the 4-momentum is P^{μ}=m $dx^{\mu}/d\tau$, so that $d\tau = (m/E)dt$ where $E = P^0 = (p_2+m_2)^{1/2}$ denotes the energy, and p, the magnitude of the 3-momentum. It follows from the geodesic equation of motion that $p \propto 1/a(t)$, so that

$p(t) = [a(t_f)/a(t)]p_f$, where p_f designates the momentum at some reference time t_f.

Thus $$\int_{t_i}^{t_f} H(\tau)d\tau = \int_{a(t_i)}^{a(t_f)} \frac{m\,da}{\sqrt{m^2a^2 + p^2a(t_f)}} = F(\gamma_f) - F(\gamma_i) \leq F(\gamma_f)$$

where $t_i < t_f$ is some initial moment.
Note that:

$$F(\gamma) = \frac{1}{2}\ln\left(\frac{\gamma+1}{\gamma-1}\right) \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\nu^2}}$$

γ is the Lorentz factor, and $v_{rel} = p/E$ is the observer's speed relative to the comoving particles.

For any non-comoving observer, γ > 1 and F(γ) > 0

The expansion rate averaged over the observer world line can be defined as

$$\text{Define: } H_{\mathrm{av}} = \frac{1}{\tau_{\mathrm{f}} - \tau_{\mathrm{i}}} \int_{t_{\mathrm{i}}}^{\tau_{\mathrm{f}}} H(\tau) d\tau.$$

Assuming that $H_{av} > 0$ and using the first equation, it follows that

$$\tau_{\mathrm{f}} - \tau_{\mathrm{i}} \leq \frac{F(\gamma_{\mathrm{f}})}{H_{\mathrm{av}}}.$$

This implies that any non-comoving past-directed timelike geodesic satisfying the condition $H_{av} > 0$, must have a finite proper length, and so must be past-incomplete.

There is no appealing to homogeneity and isotropy in an arbitrary space-time. Imagine that the universe is filled with a congruence of comoving geodesics, representing test particles and consider a non-comoving geodesic observer described by a world line $x_\mu(\tau)$

Let u_μ and v^μ designate the 4-velocities of test particles and the observer.
Then the Lorentz factor of the observer relative to the particles is

$$\gamma = u_\mu \nu^\mu$$

To characterize the expansion rate in general space-time, it suffices to focus on test particle geodesics that cross the observer's world line. Consider two such geodesics encountering the observer at times τ and $\tau + \Delta\tau$.
Define the parameter

$$H = \frac{d}{d\tau} F(\gamma(\tau))$$

with $F(\gamma) = 1/\gamma$, and γ defined by

$$H = \lim_{\Delta\tau \to 0} \frac{\Delta u_r}{\Delta r}$$

Clearly, $F(\gamma) > 0$, and the argument goes through as before.

In general relativity, a timelike congruence in a four-dimensional Lorentzian manifold can be interpreted as a family of world lines of certain ideal observers in our spacetime.

A rigorous formulation of the BGV theorem is now possible.
Let λ be a timelike or null geodesic maximally extended to the past, and let C be a timelike geodesic congruence defined along λ.

A universe that has been expanding on average throughout its history cannot be infinite in the past but must have a beginning.

If the expansion rate of C averaged along λ is positive, then λ must be past-incomplete.

XXXIV. Some Historical Models of Cosmology

Examples: Simulations of the Trajectory to the Moon and Back

Kepler Planetary Models
I. Simple Lunar Trajectories: Kepler's Elliptical Model (Planar Point Mass)
II. The Patched Conic Section Approximation for Finding a Lunar Trajectory

Newton's Planetary Models
IA. Apollo Free Return Trajectory: 3 Body Sim for CSM to the Moon & Back

Astronomy Glossaries
https://lambda.gsfc.nasa.gov/product/suborbit/POLAR/cmb.physics.wisc.edu/tutorial/glossary.html

https://ecuip.lib.uchicago.edu/multiwavelength-astronomy/glossary/glossary.html

I. Simple Lunar Trajectories: Kepler's Elliptical Model (Planar Point Mass)

This Section on Kepler is shown for historical interest. Newton's Dynamics is used in all the following Sections

Kepler's E Model (Planar Point Mass 2 Body): See the **Glossary** and **Figures** in **last two pages** of this Study

Convert Cartesian Ellipse Eq. in (x,y) to polar (r,v) coordinates
Ellipse is relative to the **focus**

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad r(\nu,e) := \frac{a\cdot\left(1-e^2\right)}{1+e\cdot\cos\left(\nu-\frac{\pi}{2}\right)}$$

$x(a,\theta) := a\cdot\cos(\theta)$ and $y(b,\theta) := b\cdot\sin(\theta)$

$0 \le t < 2\pi$ $\quad e = \frac{c}{a}$ $\quad r(x,y) := \sqrt{x^2+y^2}$ and $\theta(x,y) := \mathrm{atan}\left(\frac{y}{x}\right)$

For the moon

$e_m := .0549 \quad d_m := 384400\mathrm{km} \quad d_{ap} := 406603\mathrm{km} \quad m_m := 7.347\cdot10^{22}\mathrm{kg} \quad a_m := \frac{d_{ap}}{1+e} \quad \mu := 3.986\cdot10^5\,\frac{\mathrm{km}^3}{\mathrm{sec}^2}$

For the Earth: Mass $m_e := 5.972\cdot10^{24}\mathrm{kg}$ $\qquad$ Note: a and b are distances from the center, c

The parameter e is known as the eccentricity. The value of this parameter defines the shape of our orbit. Depending on the value of e there are four kinds of shapes (conic sections), which means there are four kinds of orbits: circle, ellipse, parabola, and hyperbola, for e = 0, < 1, = 1, and > 1, respectively.

$H \equiv 1 \quad e_e := .6 \quad e_c := 0 \quad e_h := 2 \quad e_p := 1.000 \quad \theta := 0, 0.01 .. 2\pi \qquad G\cdot m_e = 3.985\times10^{14}\,\frac{m^3}{s^2}$

Basics from Newton's Laws: Energy, Momentum, Parameters of Ellipse $\qquad r_o := 300\mathrm{km} \quad \phi_o := 0$

$$\mathrm{Energy}(v,r) := \frac{v^2}{2} - \frac{\mu}{r} \qquad h\left(v_o,r_o,\phi_o\right) := r_o\cdot v_o\cdot\cos\left(\phi_o\right) \qquad h = r^2\cdot v\nu^2 \qquad h_u(p) := \sqrt{\mu\cdot p}$$

$$p\left(v_o\right) := \frac{h\left(v_o,r_o,\phi_o\right)^2}{\mu} \qquad a\left(v_o\right) := \frac{-\mu}{\mathrm{Energy}\left(v_o,r_o\right)} \qquad e_{traj}(v,a) := \sqrt{1-\frac{p(v)}{a}}$$

Period of Moon Sat Orbit

$$T := 2\cdot\pi\cdot\sqrt{\frac{{a_m}^3}{G\cdot m_e}} = 99.98\cdot\mathrm{hr}$$

$$r_h(\theta,e) := \frac{H}{1+e\cdot\cos\left(\theta+\frac{\pi}{2}\right)}$$

If we can Solve for Eccentric Anomaly, E, we get Time of Flight, TOF, t - T

$$\cos(\nu) = \frac{p-r}{e\cdot r} \qquad \nu(p,r,e) := \mathrm{acos}\left(\frac{p-r}{e\cdot r}\right)$$

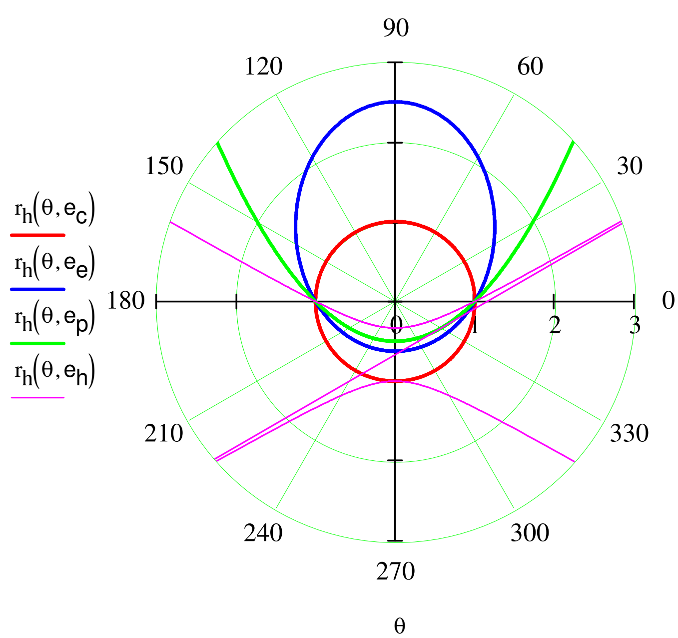
Recursion for Eccentric Anomaly, M & E (Deg)

mean anomaly M (in deg (0<= M<=360)

$$MA\left(M_o,t,t_o\right) := M_o + \sqrt{\frac{\mu}{{a_m}^3}}\cdot\left(t-t_o\right)$$

```
EcA(e,M,dp) := | mx_it ← 30
               | i ← 0
               | K ← π/180
               | del ← 10^(−dp)
               | m ← M/360
               | m ← 2·π(m − floor(m))
               | E ← m  if e < 0.8
               | E ← π  otherwise
               | F ← E − e·sin(m) − m
               | while |F| > del ∧ i < mx_it
               |    | E ← E − F/(1 − e·cos(E))
               |    | F ← E − e·sin(E) − m
               |    | i ← i + 1
               | E ← E/K
```

Plot of Conic Orbits: c, e, p, h



Find E and ϕ In Degrees

t/T: $\frac{27}{360} = 0.075 \qquad EcA\left(e_m,27,5\right) = 28.501 \qquad \phi(e,EcA) := 90 - \frac{180\cdot\mathrm{atan2}\left(\sqrt{1-e^2}\cdot\sin(EcA),\cos(EcA)-e\right)}{\pi}$

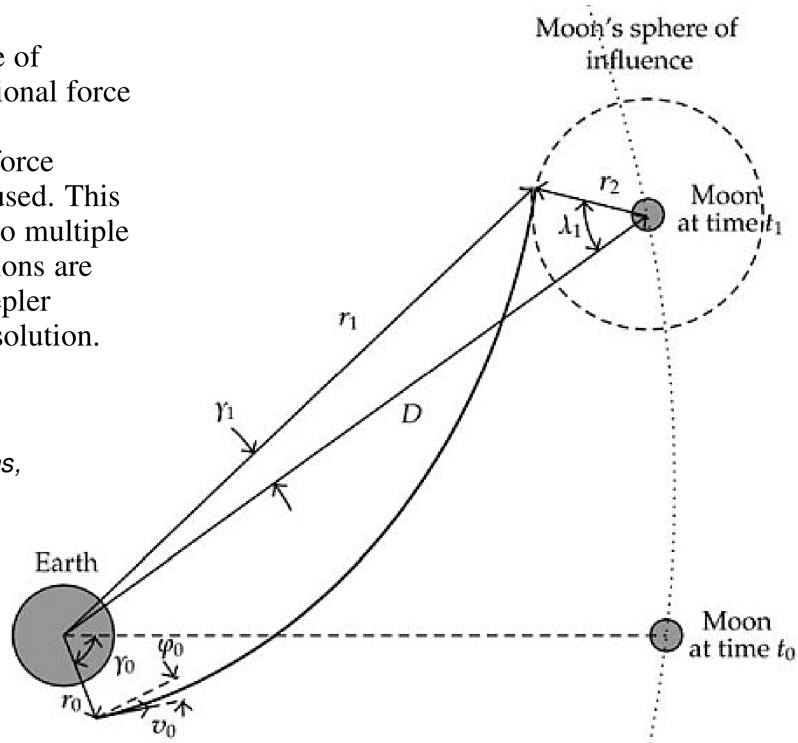
$\phi(0.977, 48.43418\cdot\mathrm{deg}) = 153.029$

The Patched Conic Section Approximation for Finding a Lunar Trajectory

The Patched Conic Method is an Approximation for finding a trajectory by dividing space between the sphere of influence (SOI) of the earth, Lunar Earth Orbit (LEO) and the SOI region of the moon.

When the spacecraft is within the sphere of influence of the moon, only the gravitational force between the spacecraft and the moon is considered, otherwise the gravitational force between the spacecraft and the earth is used. This reduces a complicated n-body problem to multiple two-body problems, for which the solutions are the well-known conic sections of the Kepler orbits. Below is an example composite solution.

See for Example:
Optimal Two-Impulse Trajectories with Moderate Flight Time for Earth-Moon Missions,
Sandro da Silva Fernandes
Mathematical Problems in Engineering
Vol. 2012, Article ID 971983,
or
Bate, R. R., D. D. Mueller, and J. E. White,
Fundamentals of Astrodynamics



Rather than dealing with large powers of 10, we can use **Astronomical Units**, for distance, velocity, time: AU, VU, TU. Where AU is the mean distance of the earth to the sun and DU is the radius of the earth. TU is the time unit. Then the velocity unit, (VU) is equal to DU/TU.

$DU := 6378.145km$ $\quad AU := 1.496 \cdot 10^8 km$ $\quad kmps := \frac{km}{s}$ $\quad VU := 7.905368kmps$ $\quad TU := 806.8s$ $\quad D := d_m$

Laplace's Equation for Moon's Sphere of Influence:
this is about 1/6 of the distance, D, to the moon

$R_{sif} := D \cdot \left(\frac{m_m}{m_e}\right)^{0.4}$ $\quad R_s := 66300km$ $\quad R_s = 10.395 \cdot DU$

The conic patched problem for finding a trajectory can be stated as follows:

Given: Initial rocket launch conditions in the earth's sphere of Influence, that is, initial position, velocity, flight path angle, and phase angle: $\mathbf{r_0}$**,** $\mathbf{v_0}$ $\mathbf{\phi_0}$**, and** $\mathbf{\gamma_0}$,

The three quntities $\mathbf{r_0}$**,** $\mathbf{v_0}$ $\mathbf{\phi_0}$ will give us initial energy and anglular momentum.

Find: Arrival conditions at moon's Sphere of Influence: $\mathbf{r_1}$**,** $\mathbf{v_1}$ $\mathbf{\phi_1}$**,** $\mathbf{\lambda_1}$**.** $\quad$ $\mathbf{r_0}$**,** $\mathbf{v_0}$ $\mathbf{\phi_0}$**, and** $\mathbf{\lambda_1}$

The problem with assigning these initial points is that they may not give a satisfactory solution to match the arrival conditions. Our strategy is to use the arrival ange λ_1 to the moon's SOI as one of the independent condition .

Given the 3 initial conditions and one arrival condition as our **independent variables:**
These will move us into the radius of the moon's sphere of influence. Some trial and error may still be required.

EXAMPLE: See Bate, R. R., D. D. Mueller, and J. E. White, *Fundamentals of Astrodynamics*

Solution: Select the **Apollo 11 Flight Conditions** for initial conditions: $\mathbf{r_0}$, $\mathbf{v_0}$, $\boldsymbol{\phi_0}$ and $\boldsymbol{\lambda_1}$.

Given: $r_0 := DU + 334km$ $v_0 := 10.6kmps$ $\phi_0 := 0deg$ A reasonable angle to arrive at moon $\lambda_1 := 30deg$

Find: $\mathbf{r_1}$, $\mathbf{v_1}$, $\boldsymbol{\phi_1}$, $\boldsymbol{\gamma_1}$ (the last symbol, γ, is the Greek letter gamma, the Arrival Phase Angle at the Moon)

Initial Energy and Angular Momentum are $Energy(v_0, r_0) = -0.011 \cdot VU^2$ $h_0 := h(v_0, r_0, \phi_0) = 1.441 \cdot \frac{DU^2}{TU}$

$D = 60.268 \cdot DU$ By the Law of Cosines: $r_1(\lambda_1) := \sqrt{D^2 + R_s^2 - 2D \cdot R_s \cdot \cos(\lambda_1)}$ $r1 := r_1(\lambda_1) = 51.529 \cdot DU$

From Law of Conservation of Energy and Momentum: $E_0 := Energy(v_0, r_0)$ $E_0 = -0.011 \cdot \frac{DU^2}{TU^2}$ $h_1 := h_0$

$v_1(r_1) := \sqrt{2 \cdot \left(E_0 + \frac{\mu}{r_1}\right)}$ $v1 := v_1(r1) = 0.128 \cdot VU$ $v1 := 0.1296VU$ $\phi_1 := acos\left(\frac{h_1}{r1 \cdot v1}\right)$ $\phi_1 = 77.542 \cdot deg$

In order to calculate the **Time of Flight**, TOF, to the moon's SOI, we need to Find:

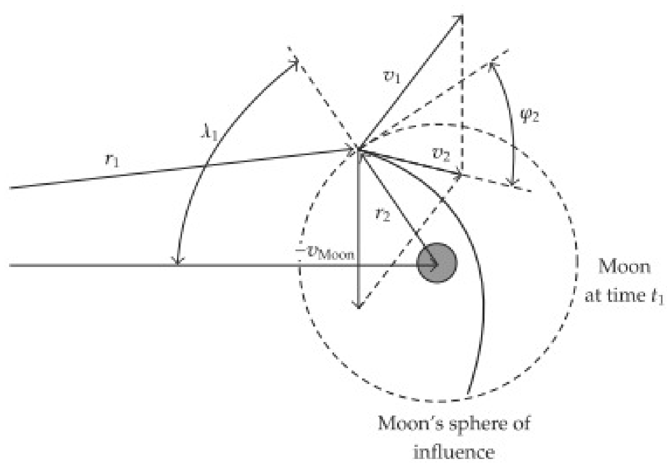
p, a, e, $\mathbf{E_0}$ and $\mathbf{E_1}$ for the Geocentric Trajectory.

$p := \frac{h_0^2}{\mu} = 2.075 \cdot DU$ $a := \frac{-\mu}{2\,Energy(v_0, r_0)}$ $e := \sqrt{1 - \frac{p}{a}}$ $e = 0.977$ $\nu_1 := \nu(p, r1, e)$ $\nu_1 = 2.956$

$\gamma_1 := asin\left(\frac{R_s}{r1} \sin(\lambda_1)\right) = 5.789 \cdot deg$ $a = 44.698 \cdot DU$ since: $\nu_0 := 0$ $EcA_0 := 0$ $EcA_1 := acos\left(\frac{e + \cos(\nu_1)}{1 + e \cdot \cos(\nu_1)}\right)$

$EcA_1 = 1.728$ $TOF := \sqrt{\frac{a^3}{\mu}} \cdot \left[\left(EcA_1 - e \cdot \sin(EcA_1)\right) - \left(EcA_0 - e \cdot \sin(EcA_0)\right)\right]$ $TOF = 51.132 \cdot hr$

We can use the same procedure at the moon (Selenocentric).
See Section XVI for the Newtonian Gravitational Solution for the Lunar Trajectory.
We need to determine the values of v1 and Rs in units based on the moon's gravitational attraction parameters.
The Angular Velocity of the Moon (ω_m) in its orbit is



: $\omega_m := 2.649 \cdot 10^{-6} \frac{rad}{s}$ $\omega_m := 2.137 \cdot 10^{-3} \frac{1}{TU}$ $\gamma_0 := \nu_1 - \nu_0 - \gamma_1 - \omega_m \cdot TOF$ $\gamma_0 = 135.637 \cdot deg$

$v_{1m} := 1.024kmps$ $\mu_m := 4093 \frac{km^3}{s^2}$ $v_m := 1.018kmps$ Then $v_{2m} := 1.198kmps$

$\varepsilon_2 := 5.68deg$ $e_m := 2.078$ $r_p := 4105km$ $h_p := 2367km$ $R_s = 10.395 \cdot DU$

$\mu_m := 4.903 \cdot 10^3 \cdot \frac{km^3}{s^2}$

Time of Flight

Develop an algorithm to Calculate Time of Flight

$$\mathrm{TOF_{alg}}(v_0, r_0, \phi_0, \lambda_1) := \left| \begin{array}{l} h_0 \leftarrow r_0 \cdot v_0 \cdot \cos(\phi_0) \\ p \leftarrow \dfrac{h_0^{\,2}}{\mu} \\ E_0 \leftarrow \mathrm{Energy}(v_0, r_0) \\ EcA_0 \leftarrow 0 \\ a \leftarrow \dfrac{-\mu}{2E_0} \\ e \leftarrow \sqrt{1 - \dfrac{p}{a}} \\ r1 \leftarrow \sqrt{D^2 + R_s^{\,2} - 2D \cdot R_s \cdot \cos(\lambda_1)} \\ \nu_1 \leftarrow \nu(p, r1, e) \\ EcA_1 \leftarrow \mathrm{acos}\left(\dfrac{e + \cos(\nu_1)}{1 + e \cdot \cos(\nu_1)}\right) \\ TOF \leftarrow \dfrac{\sqrt{\dfrac{a^3}{\mu}} \cdot \left[\left(EcA_1 - e \cdot \sin(EcA_1)\right) - \left(EcA_0 - e \cdot \sin(EcA_0)\right)\right]}{hr} \\ A \leftarrow (TOF \;\; e)^T \end{array} \right.$$

This gives a different value

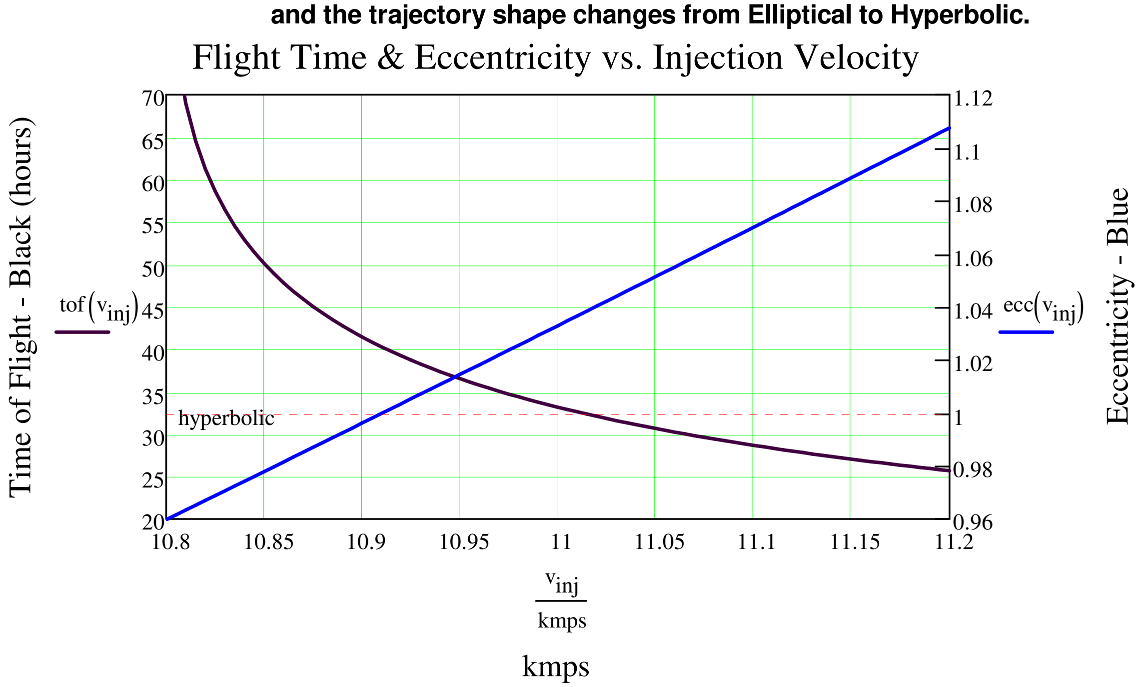
$v_0 = 10.846 \cdot \mathrm{kmps}$ $\qquad \mathrm{TOF_{alg}}(v_0, r_0, \phi_0, \lambda_1) = \begin{pmatrix} 51.132 \\ 0.977 \end{pmatrix}$

$\mathrm{tof}(v_0) := \mathrm{TOF_{alg}}(v_0, r_0, \phi_0, \lambda_1)_0$ $\qquad \mathrm{ecc}(v_0) := \mathrm{TOF_{alg}}(v_0, r_0, \phi_0, \lambda_1)_1$

Initial Conditions: $r_0 = 1.05 \cdot DU$ $\qquad \mathrm{Alititude} := r_0 - 1DU = 318.907 \cdot km$ $\qquad \phi_0 = 0$

$\mathrm{hyperbolic} := 1$ $\qquad v_{inj} := 10.8\mathrm{kmps}, 10.805\mathrm{kmps} .. 11.2\mathrm{kmps}$

Note: As the velocity increases above the minimum 10.8 kmps, the Time of Flight decreases and the trajectory shape changes from Elliptical to Hyperbolic.



Polar Plot of the Solution for the Patched Conic Lunar Approximation

$\nu := -90.002deg, -90.001deg .. 41deg$ $\chi := 39.5deg, 39.501deg .. 360deg$ $r(\nu) := \dfrac{a\cdot(1-e^2)}{1+e\cdot\cos(\nu+\gamma_0)}$

Note: λ_1 is not = 30 deg $\varphi := 33deg$ $Earth(\theta) := 1.5\sin(\theta+\varphi)$ $\theta := 0, 0.001 .. 2\pi$

$r_m := 82$ $a_m := 1.5$ $r_{moon}(\theta,\varphi) := r_m\cdot\cos(\theta-\varphi) + \sqrt{a_m^{\ 2} - r_m^{\ 2}\sin(\theta-\varphi)^2}$

Radius of Moon Sphere of Influence $r_{msi}(\theta,\varphi) := r_m\cdot\cos(\theta-\varphi) + \sqrt{10.4^2 - r_m^{\ 2}\sin(\theta-\varphi)^2}$

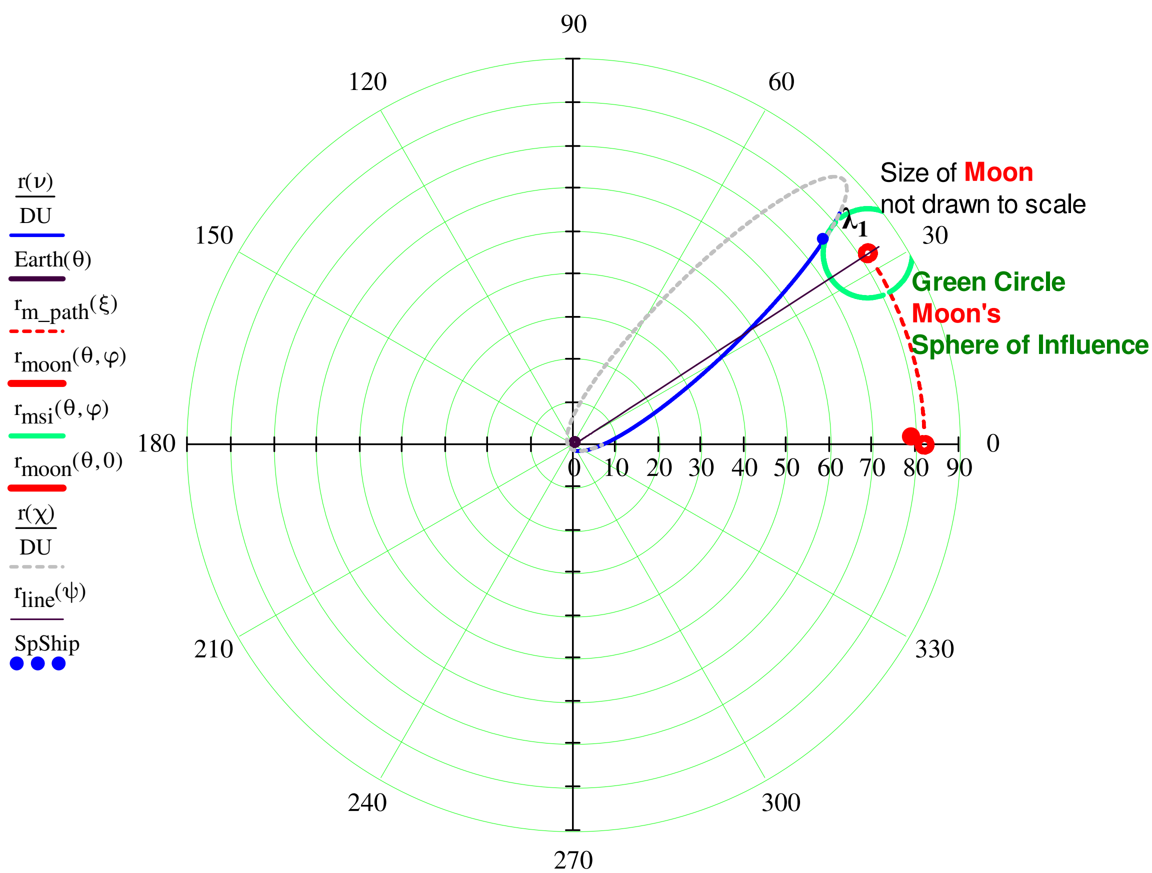
$\xi := 0.05, 0.051 .. \varphi - 0.05$ $r_{m\_path}(\xi) := r_m$ Point of Conic Patch $SpShip := 75.5$

$\psi := 0, 0.0017365 .. \varphi$ $r_{line}(\theta) := \dfrac{0.1}{\sqrt{1-(1\cdot\cos(\theta-\varphi))^2}}$ $\upsilon := 39.5deg$

Polar Plot: Geocentric Frame - Earth at the Center

From the list of functions shown on the left of the plot below:
r(v) shows the Trajectory Ellipse Conic Ptach in blue, Earth(θ) is at the center in black, $r_{moon}(\theta,\varphi)$ in red is the location of the moon at intercept $\varphi = 33^o$, $r_{msi}(\theta)$ is the circle in green of the moon's of sphere of influence, $r_{moon}(\theta,0)$ in red is the initial location of the moon at 0^o, $r_{m\_path}(\xi)$ is the dotted line path of moon from 0 to φ. r(χ) is the dotted line that shows the elliptical path back to the earth, and r_{line} is the red straight line from earth at center to the moon to show angle λ_1. SpCraft is where SpaceCraft enters the Moon's Sphere of Influence. Point of Conic Patch. Blue dot.

Patched Conic Approx. Trajectory to Moon (Red)



$e := 2.718281828459045$

$\nu, \theta, \xi, \theta, \theta, \theta, \chi, \psi, \upsilon$

IA. Apollo Free Return Trajectory: Simulation for CSM to Moon & Back

Trajectory Model: 3-Body (Earth, Moon, Spacecraft) 2D Planar Point Mass with **Earth at Center**

This 3 body gravitational solution for the FRT uses the Mathcad Differential Equation Solving Methodology discussed: arXiv:1504.07964
"***Motion of the planets: the calculation and visualization in Mathcad***", Valery Ochkov, Katarina Pisa

The aborted Apollo 13 mission was the only mission to actually turn around the Moon in a free-return trajectory.

Solve the Gravitational and Dynamics Equations for Earth, Moon, & CSM Trajecto

$kg := 1$ $m := 1$ $s := 1$ $N := 1$ $s := 1$ $min := 60s$ $hr := 3600s$ $kgf := 9.80665N$

$km := 1000m$ $kmps := km$ $kph := \frac{km}{hr}$ $mph := 0.447 \cdot 10^{-3} kmps$ $G := 6.67384 \cdot 10^{-11} \cdot \frac{N \cdot m^2}{kg}$

Run Simulation for 160 hrs **Apollo 11 Orbit 77 hrs**

$FRAME := 999$ $n_{ode} := 20000$ $n := 999$ $n_{plot} := 10000$ $t_{end} := \frac{160hr}{n+1} \cdot (FRAME + 1)$ $t_{orb} = 81.44\ hr$

Time of Flight (TOF) = t_{orb}

Trajectory to Moon's Sphere of Influence

Initial x,y Velocity CSM **Radius of Earth** **Apogee to Moon**

$v_{0x} := 6.811kmps$ $v_{0y} := 6.356kmps$ $v_{CSM} := 9.317kmps$ $R_e := 6370km$ $d_{m\_ap} := 405500km$

Define Gravitational and Dynamics Equations for Earth, Moon, and CSM

Mass Start position Start Velocity

$$\begin{array}{l} \text{Earth, e} \\ \text{Moon, m} \\ \text{CSM, s} \end{array} \begin{pmatrix} m_e & x_{e0} & y_{e0} & vx_{e0} & vy_{e0} \\ m_m & x_{m0} & y_{m0} & vx_{m0} & vy_{m0} \\ m_s & x_{s0} & y_{s0} & vx_{s0} & vy_{s0} \end{pmatrix} := \begin{pmatrix} 5.972 \cdot 10^{24} kg & 0m & 0m & 0kph & 0kph \\ 7.347 \cdot 10^{22} kg & d_{m\_ap} & 0km & 0kmps & 0.97kmps \\ 13600kg & R_e + 100km & R_e - 100km & v_{0x} & v_{0y} \end{pmatrix}$$

Given **Solve Set of Differential Guidance Equations for 3 Body Problem of Earth, Moon, and CSM**

$x_e(0) = x_{e0}$ $x_e'(0) = vx_{e0}$ $y_e(0) = y_{e0}$ $y_e'(0) = vy_{e0}$

$$m_e \cdot x_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (x_m(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (x_s(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3}$$

$$m_e \cdot y_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (y_m(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (y_s(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3}$$

$x_m(0) = x_{m0}$ $x_m'(0) = vx_{m0}$ $y_m(0) = y_{m0}$ $y_m'(0) = vy_{m0}$

$$m_m \cdot x_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (x_e(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_m \cdot m_s \cdot (x_s(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3}$$

$$m_m \cdot y_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (y_e(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G\, m_m \cdot m_s \cdot (y_s(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3}$$

$x_s(0) = x_{s0}$ $x_s'(0) = vx_{s0}$ $y_s(0) = y_{s0}$ $y_s'(0) = vy_{s0}$

$$m_s \cdot x_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (x_e(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (x_m(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3}$$

$$m_s \cdot y_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (y_e(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (y_m(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3}$$

IA. Free Return Trajectory: 3 Body Sim for CSM to the Moon & Back

Trajectory Model: 3-Body (Earth, Moon, Spacecraft) 2D Planar Point Mass with **Earth at Center**

Differential Equation Solver

$$\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \end{pmatrix}, t, t_{end}, n_{ode}\right]$$

Initial Velocity (km/s) of CSM at an Altitude of 141 km:

$$\sqrt{v_{0x}^2 + v_{0y}^2} = 9.316 \cdot \text{kmps}$$

Distance to the Center of the Moon

$$d_m(t) := \left[\sqrt{\left(x_m(t) - x_s(t)\right)^2 + \left(y_m(t) - y_s(t)\right)^2}\right] \qquad \frac{d_m(t_{orb})}{R_m} = 1.619$$

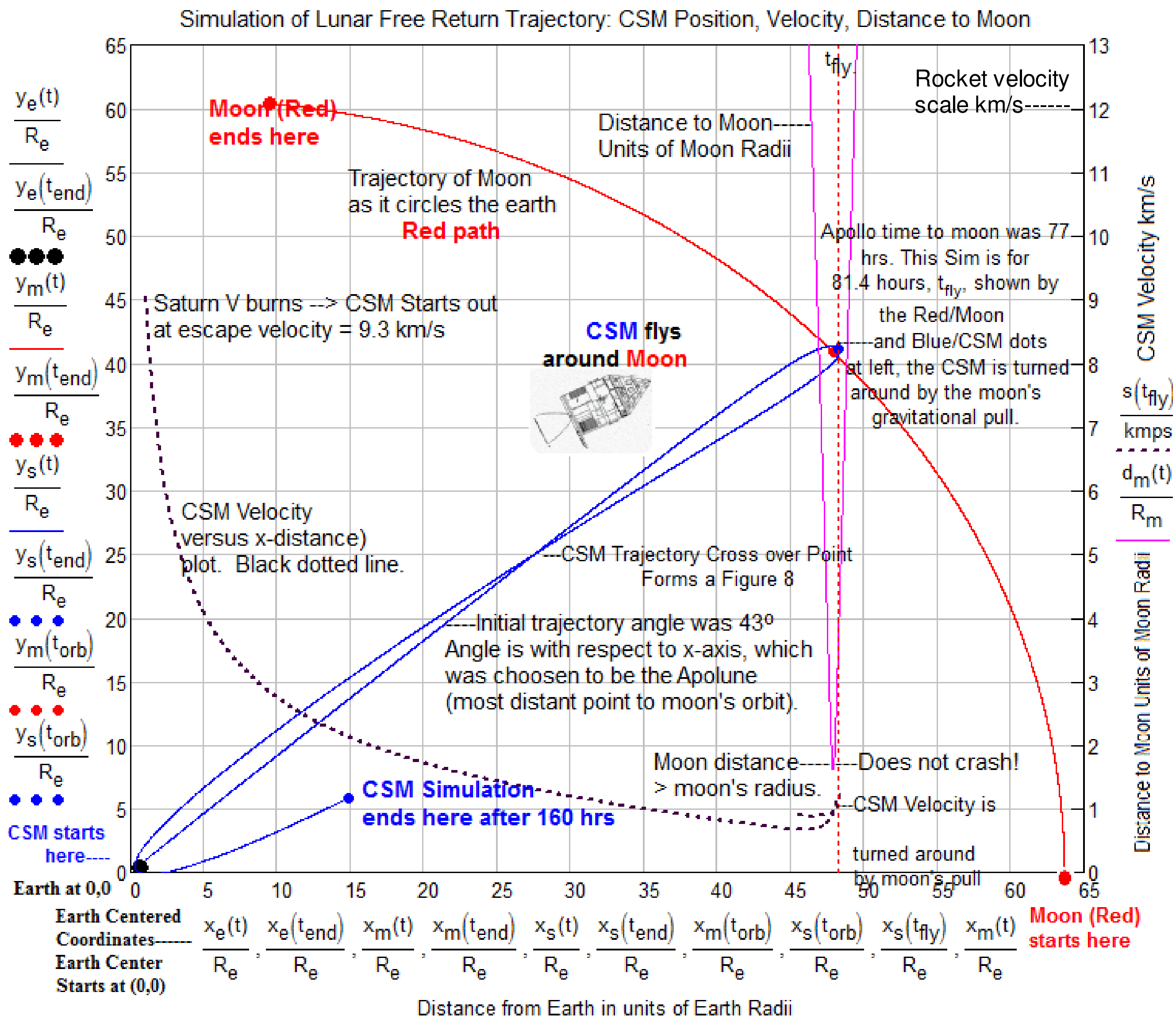
time t_{fly}, just beyond lunar fly by time $\quad$ **t** $_{flydot}$

$$t := 0, \frac{t_{end}}{n_{plot}} .. t_{end} \qquad t_{fly} := 0, \frac{t_{end}}{n_{plot}} .. 90hr \qquad t_{fly.} := \frac{x_s(t_{orb})}{R_e}$$

$$vx_s(t) := \frac{d}{dt}x_s(t) \qquad vy_s(t) := \frac{d}{dt}y_s(t) \qquad s(t) := \sqrt{vx_s(t)^2 + vy_s(t)^2}$$

Finding a Free Return Trajectory (FRT) is a little tricky. First, the trajectory must catch the moon at the exact place and time as travels around the earth and then after being swing around by the moon's gravity it must swing back and catch the earth in such a way as to go into earth orbit. This can present a problem for the Differential Equation Solver. This is a three body problem. A change in the CSM's trajectory is influenced by the pull the moon, which in turn is affected by the pull of the earth. The solver can easily fail to converge on a solution. A change in angle by 10 degrees can result in a large change in orbit time of 4.5 days. We also must check that CSM does not crash into moon.

Below is a plot of our FRT solution for the Apollo Trajectory. It shows the CSM's x,y position and velocity from earth to moon and back. .Note the figure 8 orbit of this Free Return. The Apollo 11 flight time to the moon was 77 hours. Our simulation is for 81.4 hours. Because of instabilities, convergence problems, etc. some trial and error was required.



IB. 4-Body Sim of Apollo Free Return Trajectory: CSM to Moon & Back

Trajectory Model: 4-Body (Earth, Moon, Sun, Spacecraft) 2D Planar Point Mass w **Earth at Center**

This Simulation Uses the Mathcad Differential Equation Solving Methodology discussed in: arXiv:1504.07964
"***Motion of the planets: the calculation and visualization in Mathcad***", Valery Ochkov, Katarina Pisačić

4-Body Reference Frame: Earth and moon are initially at 0,0 and the earth and sun are initially not moving.

$kg := 1$ $m := 1$ $s := 1$ $N := 1$ $\underset{\sim}{s} := 1$ $min := 60s$ $hr := 3600s$ $kgf := 9.80665N$

$km := 1000m$ $kmps := km$ $kph := \frac{km}{hr}$ $mph := 0.447 \cdot 10^{-3} kmps$ $n_{plot} := 10000$

Run Simulation for 115 hrs **Apollo 11 Orbit 77 hr**

$FRAME := 999$ $n_{ode} := 20000$ $n := 999$ $t_{end} := \frac{114.5hr}{n+1} \cdot (FRAME + 1)$ $t_{orb} := 58.5hr$

Time of Flight (TOF) = t_{orb}

Trajectory to Moon's Sphere of Influence Apolune

$G := 6.67384 \cdot 10^{-11} \frac{N \cdot m^2}{kg}$

Initial x,y Velocity CSM **Radius of Earth** **Apogee to Moon**

$v_{0x} := 7.58kmps$ $v_{0y} := 5.5kmps$ $R_m := 1737.4km$ $R_e := 6370km$ $d_{m\_ap} := 405500km$

$t_{end} = 114.5 hr$ $v_{CSM} := \sqrt{v_{0x}^2 + v_{0y}^2}$ $v_{CSM} = 9.365 \cdot kmps$ $d_{e\_ap} := 152 \cdot 10^6 km$

Define Gravitational and Dynamics Equations for Earth, Moon, and CSM

$$
\begin{array}{l} \text{e is Earth} \\ \text{a is Sun} \\ \text{m is Moon} \\ \text{s is CSM} \end{array}
\begin{pmatrix} m_e & x_{e0} & y_{e0} & vx_{e0} & vy_{e0} \\ m_a & x_{a0} & y_{a0} & vx_{a0} & vy_{a0} \\ m_m & x_{m0} & y_{m0} & vx_{m0} & vy_{m0} \\ m_s & x_{s0} & y_{s0} & vx_{s0} & vy_{s0} \end{pmatrix} :=
\begin{pmatrix} 5.972 \cdot 10^{24} kg & 0m & 0m & 0kph & 0kmps \\ 1.989 \cdot 10^{30} kg & -130 \cdot 10^6 km & -80 \cdot 10^6 km & 0kmps & 0kmps \\ 7.347 \cdot 10^{22} kg & d_{m\_ap} & 0km & 0kmps & 0.97kmps \\ 13600kg & R_e + 110km & R_e - 96km & v_{0x} & v_{0y} \end{pmatrix}
$$

Given **Set of Differential Guidance Equations for 4 Body Problem of Earth, Moon, and CSM**

$x_e(0) = x_{e0}$ $x_e'(0) = vx_{e0}$ $y_e(0) = y_{e0}$ $y_e'(0) = vy_{e0}$ $x_m(0) = x_{m0}$ $x_m'(0) = vx_{m0}$ $y_m(0) = y_{m0}$ $y_m'(0) = vy_{m0}$

$$m_e \cdot x_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (x_m(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (x_s(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_S \cdot (x_S(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_S(t))^2 + (y_e(t) - y_S(t))^2}\right]^3}$$

$$m_e \cdot y_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (y_m(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (y_s(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_S \cdot (y_S(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_S(t))^2 + (y_e(t) - y_S(t))^2}\right]^3}$$

$$m_m \cdot x_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (x_e(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_m \cdot m_s \cdot (x_s(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_m \cdot m_S \cdot (x_S(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_S(t))^2 + (y_m(t) - y_S(t))^2}\right]}$$

$$m_m \cdot y_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (y_e(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G m_m \cdot m_s \cdot (y_s(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3} + \frac{G m_m \cdot m_S \cdot (y_S(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_S(t))^2 + (y_m(t) - y_S(t))^2}\right]}$$

$x_s(0) = x_{s0}$ $x_s'(0) = vx_{s0}$ $y_s(0) = y_{s0}$ $y_s'(0) = vy_{s0}$ $x_S(0) = x_{S0}$ $x_S'(0) = vx_{S0}$ $y_S(0) = y_{S0}$ $y_S'(0) = vy_{S0}$

$$m_s \cdot x_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (x_e(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (x_m(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_S \cdot (x_S(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

$$m_s \cdot y_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (y_e(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (y_m(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_S \cdot (y_S(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

$$m_S \cdot x_S''(t) = \frac{G \cdot m_S \cdot m_e \cdot (x_e(t) - x_S(t))}{\left[\sqrt{(x_e(t) - x_S(t))^2 + (y_e(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_m \cdot (x_m(t) - x_S(t))}{\left[\sqrt{(x_m(t) - x_S(t))^2 + (y_m(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_s \cdot (x_s(t) - x_S(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

$$m_S \cdot y_S''(t) = \frac{G \cdot m_S \cdot m_e \cdot (y_e(t) - y_S(t))}{\left[\sqrt{(x_e(t) - x_S(t))^2 + (y_e(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_m \cdot (y_m(t) - y_S(t))}{\left[\sqrt{(x_m(t) - x_S(t))^2 + (y_m(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_s \cdot (y_s(t) - y_S(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

IB. 4-Body Sim of Apollo Free Return Trajectory: CSM to Moon and Back

Trajectory Model: 4-Body (Earth, Moon, Sun, Spacecraft) 2D Planar Point Mass w **Earth at Center**

Plot for Sim of 4-Body Free Return Traj: CSM to Moon and Back

Differential Equation Solver

Earth, Moon, Space Craft, Sun

$$\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \\ x_a \\ y_a \end{pmatrix} := \mathrm{Odesolve}\left[\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \\ x_a \\ y_a \end{pmatrix}, t, t_{end}, n_{ode}\right]$$

Initial Velocity (km/s) of CSM at an Altitude of 141 km:

$$\sqrt{v_{0x}{}^2 + v_{0y}{}^2} = 9.365 \cdot \mathrm{kmps}$$

time t_{fly}, just beyond lunar fly by time **t flyby dot**

$$t := 0, \frac{t_{end}}{n_{plot}} .. t_{end} \qquad t_{fly} := 0, \frac{t_{end}}{n_{plot}} .. 160\mathrm{hr} \qquad t_{fly.} := \frac{x_s(t_{orb})}{R_e}$$

$$vx_s(t) := \frac{d}{dt}x_s(t) \qquad vy_s(t) := \frac{d}{dt}y_s(t) \qquad s(t) := \sqrt{vx_s(t)^2 + vy_s(t)^2}$$

Distance from Earth to Moon

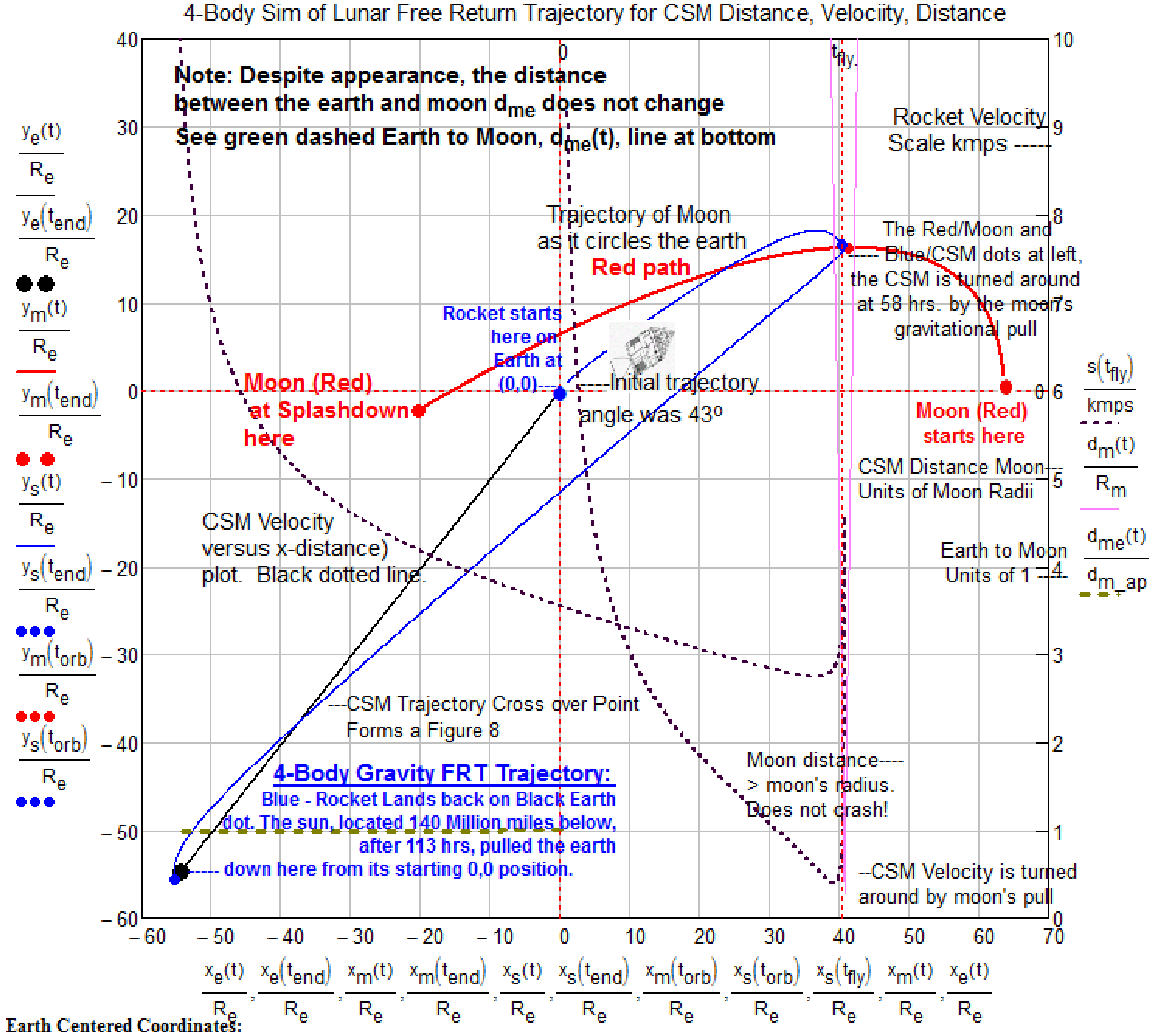
$$d_{me}(t) := \left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]$$

Distance to the Center of the Moon

$$d_m(t) := \left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]$$

$$\frac{d_m(t_{orb})}{R_m} = 3.242$$

4-Body Sim of Lunar Free Return Trajectory for CSM Distance, Velociity, Distance



Earth Centered Coordinates: Center of Earth Starts at (0,0), but gravitational pull of sun, 94 million miles below-left of earth pulls the earth down & left from 0,0 so it ends at black dot above. The rocket (blue dot)lands back on earth 114 hours after launch.

Note: The radial velocity of the earth around the sun is 1º every 365 days or 1/365º per day. Our sim runs 114 hrs or 114/24 days. This results in (1/365º) x 114/24 or 7.5º. For the purpose of our illustration, we will ignore this added complexity. Think of this as a rotating reference frame, such as our experience of us living on a rotating earth